TOPOLOGICAL HOCHSCHILD HOMOLOGY AND CYCLIC K-THEORY

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Abstract. These are notes for a talk at the MFO Arbeitstagung 'Topological Cyclic Homology' in April 2018. We give a definition of topologoical cyclic homology and cyclic K-theory for stable ∞-categories and relate the two notions by an enhancement of the usual cyclotomic trace.

Contents

1. Introduction

Most of the material presented here is based on ideas of Kaledin, Keller and Blumberg-Mandell. An alternative definition of THH and the trace for stable ∞-categories has been given by Ayala, Mazel-Gee and Rozenblyum.

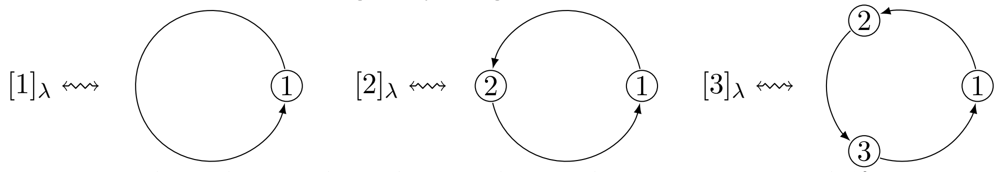
Not in this note: symmetric monoidal, spectrally enriched.

2. Cyclic, paracyclic and epicyclic objects

We begin this note by recalling Connes cyclic category Λ. We first define a related category Λ_∞, the paracyclic category. It is defined as full subcategory of the category of ordered sets with an order preserving $\mathbb{Z}$-action in which morphisms are non-decreasing equivariant maps. Then Λ_∞ consists of those objects which are isomorphic to $\frac{1}{n}\mathbb{Z}$ with the obvious ordering and the $\mathbb{Z}$-action by addition of integer.

We denoted the object $\frac{1}{n}\mathbb{Z}$ also by $[n]_\Lambda \in \Lambda_\infty$ and by definition every object in Λ_∞ is equivalent to one of those.

There is a canonical functor $\Delta \to \Lambda_\infty$ which sends a non-empty linearly ordered set S to the set $\mathbb{Z} \times S$ with lexicographic ordering and $\mathbb{Z}$ action by addition in the left factor. For every natural number $k \geq 1$ we define a category Λ_k by identifying certain morphisms in Λ_∞, namely the quotient by the relation $f \sim f + k$ for $k \in \mathbb{Z}$ (here the $\mathbb{Z}$-action is written additively). Then the cyclic category is $\Lambda_1 =: \Lambda$. The object of Λ (resp. Λ_k) corresponding to $\frac{1}{n}\mathbb{Z}$ is written as $[n]_\Lambda$ (resp. $[n]_{\Lambda_k}$). One should think of Λ as consisting of cyclic graphs:



For every such cyclic graph with n nodes we have an associated 'free category' $\mathbb{T}_n \in \mathrm{Cat}$ and then a map $[n]_\Lambda \to [m]_\Lambda$ in Λ corresponds to a functor $\mathbb{T}_n \to \mathbb{T}_m$ of these categories that is a map of degree one of the circle after geometric realization. Really the poset $\frac{1}{n}\mathbb{Z}$ with its $\mathbb{Z}$-action should be considered as the universal cover of such a graph/category. In particular this assignment gives us a functor

$$\Lambda \to \mathrm{Cat} \qquad [n]_\Lambda \mapsto \mathbb{T}_n \ . \tag{1}$$

In a choice free way we can write this functor as sending a poset $P \in \Lambda$ with $\mathbb{Z}$-action to the quotient category $\mathbb{T}_P := P/\mathbb{Z}$ where the poset P is considered as a category and the quotient is taken in the category of categories.

The epicyclic category $\tilde{\Lambda}$ is by definition the subcategory of Cat consisting of categories isomorphic to $\mathbb{T}_n$ for some n and functors that are surjective on objects (equivalently: essentially surjective). Here we consider Cat as a 1-category neglecting the existence of natural transformations (this is just for the combinatorics, the default is of course to consider it as a 2-category). We denote the object corresponding to $\mathbb{T}_n$ in $\tilde{\Lambda}$ by $[n]_{\tilde{\Lambda}}$. There is by construction a functor $\Lambda \to \tilde{\Lambda}$ and this functor is faithful and essentially surjective, but not full. The main difference is that in $\tilde{\Lambda}$ we have 'degree k-maps' $[kn]_{\tilde{\Lambda}} \to [n]_{\tilde{\Lambda}}$ for $k > 0$ that do not exist in Λ. The epicyclic category was introduced by Goodwillie in an unpublished letter to Waldhausen 1987, see also the discussion in [BFG94] for a generators and relations description of $\tilde{\Lambda}$. By definition the functor (1) extends to a functor

$$\tilde{\Lambda} \to \mathrm{Cat} \qquad [n]_{\tilde{\Lambda}} \mapsto \mathbb{T}_n \ .$$

Construction 2.1. Similar to the definition of Λ as a quotient of Λ_∞ by a $B\mathbb{Z}$-action[1] there is a combinatorial way to describe $\tilde{\Lambda}$ as follows. Consider the topological monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$ which is the semidirect product of the circle $\mathbb{T}$ and the *multiplicative* monoid of positive natural numbers where the latter acts on $\mathbb{T}$ by sending $n \in \mathbb{N}_{>0}$ to the degree n map $z \in \mathbb{T} \mapsto z^n \in \mathbb{T}$. Another description of the monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$ is that it is homotopy equivalent to the monoid of self maps of the circle $\mathbb{T}$ (considered as a homotopy type) consisting of maps that have positive degree.

[1]Here $B\mathbb{Z}$ denotes the category with a single object and $\mathbb{Z}$ as endomorphisms. This is a monoid object in Cat and it is not hard to see that the action of the integers on the hom sets of Λ_∞ gives an action of this monoid on Λ_∞.

There is an action of the monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$ on the paracyclic category Λ_∞. We think of $\mathbb{T} \rtimes \mathbb{N}_{>0}$ as the monoid $(B\mathbb{Z}) \rtimes \mathbb{N}_{>0}$ in Cat and define the action of $B\mathbb{Z}$ as acting by addition on the hom set (this is the action whose quotient is by definition Λ). For a natural number $n \in B\mathbb{Z} \rtimes \mathbb{N}_{>0}$ we define the action of n on Λ_∞ by sending a poset P with $\mathbb{Z}$-action to the new object $n \cdot P$ which is also P as a poset but where the action of $\mathbb{Z}$ on P is given by acting with $k \in \mathbb{Z}$ by addition of nk. In other words, we just pull back the action along the map $\mathbb{Z} \to \mathbb{Z}$ given by mutliplication with n. In terms of standard objects this functor sends $[k]_{\Lambda_\infty}$ to $[nk]_{\Lambda_\infty}$. It is then obvious that this gives an action of $B\mathbb{Z} \rtimes \mathbb{N}_{>0}$ on Λ_∞.

Definition 2.2. *For an action of an $\mathbb{E}_1$-space M on an ∞-category $\mathcal{C}$ we define the* lax quotient $\mathcal{C}_{\ell M}$ *to be the total space of the Grothendieck construction (a.k.a. unstraightening) of the associated functor $BM \to \mathrm{Cat}_\infty$.*

We can describe $\mathcal{C}_{\ell M}$ informally as follows: objects of $\mathcal{C}_{\ell M}$ are given by objects of $\mathcal{C}$. A morphism $c \to c'$ in $\mathcal{C}_{\ell M}$ consists of a pair (f, m) where $m \in M$ and f is a morphism in $\mathcal{C}$ from $m \cdot c$ to c'. By construction as a Grothendieck construction we have a coCartesian functor $\mathbb{C}_{\ell M} \to BM$. If M happens to be a group then the lax quotient is equivalent to the homotopy quotient $\mathcal{C}_{hM}$. In fact the homotopy quotient $\mathcal{C}_{hM}$ is always obtained from $\mathcal{C}_{\ell M}$ by Dwyer-Kan localizing at the set of coCartesian edges [Lur09, Corollary 3.3.4.3].

We now want to investigate the lax quotient of the action of $\mathbb{T} \rtimes \mathbb{N}_{>0}$ on the ∞-category $\mathrm{N}\Lambda_\infty$. Note that since Λ_∞ is a 1-category and the monoid $\mathbb{T} \rtimes \mathbb{N}$ is a homotopy 1-type the ∞-category $(\mathrm{N}\Lambda_\infty)_{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})}$ is a priori a 2-category. It turns out that in this case it is an ordinary category. More precisely we have:

Lemma 2.3. *There is an equivalence $(\mathrm{N}\Lambda_\infty)_{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})} \simeq \mathrm{N}\tilde{\Lambda}$.*

Proof. We have an equivalence $(B\mathbb{T})_{\ell\mathbb{N}_{>0}} \simeq B(\mathbb{T} \rtimes \mathbb{N}_{>0})$ which gives us by abstract nonsense an equivalence

$$(\mathrm{N}\Lambda_\infty)_{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})} \simeq ((\mathrm{N}\Lambda_\infty)_{\ell\mathbb{T}})_{\ell\mathbb{N}_{>0}} \ .$$

The lax orbits by $\mathbb{T}$ are equivalent to the homotopy orbits since $\mathbb{T}$ is a group. But the homotopy orbits are equivalent to $\mathrm{N}\Lambda$ by definition of Λ and the fact that the $\mathbb{Z}$-action on hom sets is free. Thus we get that

$$(\mathrm{N}\Lambda_\infty)_{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})} \simeq (\mathrm{N}\Lambda)_{\ell\mathbb{N}_{>0}} \ .$$

Now this lax quotient is a 1-category since $B\mathbb{N}_{>0}$ and $\mathrm{N}\Lambda$ are. Then he claim follows from the fact that functor of ordinary categories

$$\tilde{\Lambda} \to B\mathbb{N}_{>0}$$

which sends a morphism in $\tilde{\Lambda}$ to its degree is coCartesian and classified by the $\mathbb{N}_{>0}$-action on Λ. This fact is elementary to verify. □

A corollary of Lemma 2.3 is that the geometric realization of $\tilde{\Lambda}$ is given by the classifying space of the monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$ since Λ_∞ has contractible classifying space (see [NS17, Appendix T]) . This result was first obtained in [BFG94].

Definition 2.4. *Let $\mathcal{C}$ be an ∞-category. A functor $\mathrm{N}\Lambda^{\mathrm{op}} \to \mathcal{C}$ is called a cyclic object in $\mathcal{C}$. The geometric realization of a cyclic object $\mathrm{N}\Lambda^{\mathrm{op}} \to \mathcal{C}$ is by definition the colimit (if it exists) over the 'underlying' simplicial object*

$$\mathrm{N}\Delta^{\mathrm{op}} \to \mathrm{N}\Lambda_\infty^{\mathrm{op}} \to \mathrm{N}\Lambda^{\mathrm{op}} \to \mathcal{C}.$$

An epicyclic object is a functor $\mathrm{N}\tilde{\Lambda}^{\mathrm{op}} \to \mathcal{C}$*. The geometric realization of an epicyclic object is similarly the realization of the underlying simplicial object.*

One can show that the functor $\mathrm{N}\Delta \to \mathrm{N}\Lambda_\infty$ is final (see e.g. [NS17, Appendix T]) so that the colimit defining the geometric realization of an (epi)cyclic object is equivalent to the colimit over the associated paracyclic object $\Lambda_\infty^{\mathrm{op}} \to \Lambda \to \mathcal{C}$. The crucial fact is that for every cyclic object in $\mathcal{C}$ the geometric realization inherits a $\mathbb{T}$-action (where $\mathbb{T} = S^1$ is the circle group) and for every epicyclic object the realization inherits an action of the monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$. Let us describe where this action comes from: Restricting to the paracylic category defines functors

$$\mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}\left(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}\right)^{h\mathbb{T}} \qquad \text{and} \qquad \mathrm{Fun}(\tilde{\Lambda}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}\left(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}\right)^{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})}$$

where $\mathbb{T} \rtimes \mathbb{N}_{>0}$ acts on $\mathrm{Fun}(\tilde{\Lambda}^{\mathrm{op}}, \mathcal{C})$ by acting in the argument and where $\mathcal{D}^{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})}$ for an ∞-category $\mathcal{D}$ with $\mathbb{T} \rtimes \mathbb{N}_{>0}$-action denotes the *lax fixed points*, i.e. sections of $\mathcal{D}_{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})} \to B(\mathbb{T} \rtimes \mathbb{N}_{>0})$. One should think of $\mathrm{Fun}\left(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}\right)^{h\mathbb{T}}$ as equivariant functors and as $\mathrm{Fun}\left(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}\right)^{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})}$ as 'lax equivariant' functors where $\mathcal{C}$ carries the trivial action. The colimit gives functors

$$\mathrm{Fun}\left(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}\right)^{h\mathbb{T}} \to \mathcal{C}^{h\mathbb{T}} \simeq \mathcal{C}^{B\mathbb{T}}$$

and

$$\mathrm{Fun}\left(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}\right)^{\ell(\mathbb{T}\rtimes\mathbb{N})} \to \mathcal{C}^{\ell(\mathbb{T}\rtimes\mathbb{N}_{>0})} \simeq \mathcal{C}^{B(\mathbb{T}\rtimes\mathbb{N}_{>0})}$$

by [**?**] so that we obtain by composition the next result. [2]

Proposition 2.5. *For any ∞-category $\mathcal{C}$ that admits geometric realizations of simplicial objects, geometric realization of cyclic and epicyclic objects canonically gives functors*

$$\mathrm{Fun}(\mathrm{N}\Lambda^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}^{B\mathbb{T}} \qquad \textit{and} \qquad \mathrm{Fun}(\mathrm{N}\tilde{\Lambda}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}^{B(\mathbb{T}\rtimes\mathbb{N}_{>0})}\ .$$

For a more point set treatment of epicyclic objects see [BFG94]. Since epicyclic objects are not so well known we make a bit more explicit what it means to have an epicyclic object and how to think of the action on the geometric realization.This also shows the relation of cyclotomic structures discussed later and in [NS17]. For the remaineder of the paper the following discussion is not relevant.

By definition every epicyclic object $X : \tilde{\Lambda}^{\mathrm{op}} \to \mathcal{C}$ has an underlying cyclic object. The additional structure that exist for an epicyclic object can be formulated as follows: we can form the n-fold edgewise subdivision $\mathrm{sd}_k^* X$ which is a k-cyclic object, i.e. a functor $\Lambda_k^{\mathrm{op}} \to \mathcal{C}$ and is obtained by pullback along

$$\mathrm{sd}_k : \Lambda_k \to \Lambda \qquad P \mapsto k \cdot P$$

where $k \cdot P$ is as before the same poset P but with $\mathbb{Z}$-action multiplied by k. The object $k \cdot P$ carries a C_k-action where C_k is the cyclic group of order k. Thus we get that the subdivision gives a functor

$$\mathrm{sd}_k X^* : \Lambda_k^{\mathrm{op}} \to \mathcal{C}^{BC_k}\ .$$

Moreover this functor is BC_k-equivariant where BC_k acts on the source by the quotient of the $B\mathbb{Z}$-action on Λ_∞ and on the target by left-multiplication on BC_k using that C_k is abelian. This follows from the fact that the functor $\mathrm{sd}_k : \Lambda_k \to \Lambda^{BC_k}$ is BC_k-equivariant which is straighforward to verify. Now we take homotopy fixed

[2] TODO: Check that there is no $(-)^{\mathrm{op}}$ showing up somewhere here.

points by C_k (assuming that they exist in $\mathcal{C}$) i.e. postcompose with the functor $(-)^{hC_k}\colon \mathcal{C}^{BC_k} \to \mathcal{C}$. This functor is also BC_k-equivariant. Therefore we obtain a composition as in the upper horizontal line of the diagram

$$\begin{array}{ccccccc}
\Lambda_p^{\mathrm{op}} & \xrightarrow{\mathrm{sd}_k} & (\Lambda^{\mathrm{op}})^{BC_k} & \xrightarrow{X^{BC_k}} & \mathcal{C}^{BC_k} & \xrightarrow{_^{hC_k}} & \mathcal{C} \\
\downarrow & & & & & \nearrow & \\
\Lambda^{\mathrm{op}} & - & - & - & - & - &
\end{array}$$

which is BC_p-equivariant and therefore descends to a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$. This is just a fancy way of arguing that the diagram $(\mathrm{sd}_k^* X)^{hC_p}$ is again a cyclic object. In a point set model this can also be verified directly. The point is that the action of $\mathbb{N}_{>0}$ on Λ_∞ is precisely encoding this behaviour of the functor. Thus an extension of a cyclic object X to a epicyclic object gives rise to a natural transformation

$$X \to (\mathrm{sd}_k^* X)^{hC_k}$$

induced by the covering maps of degree k using that in $\tilde{\Lambda}$ we have that $[nk]_{hC_k} \simeq [n]$. After geometric realizing and using that $\mathrm{sd}_k : \Lambda_\infty \to \Lambda_\infty$ is final this gives a $\mathbb{T}$-equivariant map

$$\psi_k : |X| \to |\mathrm{sd}_k^* X^{hC_k}| \to |\mathrm{sd}_k^* X|^{hC_k} \simeq |X|^{hC_k}$$

where $|X|^{hC_k}$ carries the residual $\mathbb{T}/C_k \simeq \mathbb{T}$-action. Conversely it is very easy to verify that an action of $\mathbb{T} \rtimes \mathbb{N}_{>0}$ on $|X|$ is essentially given by a $\mathbb{T}$-action on $|X|$ together with such a map $\psi_k : |X| \to |X|^{hC_k}$ for every k such that these maps commute coherently (in the appropriate sense involving fixed points for the product). In this sense the action of $\mathbb{T} \rtimes \mathbb{N}_{>0}$ is essentially determined by the maps ψ_k.

3. Unstable topological Hochschild homology

Now we will use the notion of (epi)cyclic objects to give the definition of unstable topological Hochschild homology. The word 'unstable' indicates that we work in the ∞-category $\mathcal{S}$ of spaces. Later we will work in spectra to obtain topological Hochschild homology. For an ∞-category $\mathcal{D}$ we denote by $\mathcal{D}^\sim$ the maximal Kan complex inside of $\mathcal{D}$, i.e. the groupoid core.

Definition 3.1. *For a small ∞-category $\mathcal{C}$ we define a space* $\mathrm{uTHH}(\mathcal{C}) \in \mathcal{S}^{B(\mathbb{T} \rtimes \mathbb{N}_{>0})}$ *as the geometric realization of the epicyclic object*

$$\tilde{\Lambda}^{\mathrm{op}} \to \mathcal{S} \qquad [n]_{\tilde{\Lambda}} \mapsto \mathrm{Fun}\,(\mathrm{N}\mathbb{T}_n, \mathcal{C})^\sim \ .$$

Remark 3.2. One can check that for a given $\mathcal{C}$ there is an equivalence

$$\mathrm{Fun}\,(\mathrm{N}\mathbb{T}_n, \mathcal{C})^\sim \simeq \underset{c_1,\dots,c_n \in \mathcal{C}^\sim}{\mathrm{colim}} \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{C}}(c_i, c_{i+1})$$

where $c_{n+1} = c_1$. This is the formula that we generalize for stable ∞-categories in Section 6.

Now we want to compare this definition of unstable topological Hochschild homology to the usual definition using the standard cyclic Bar construction. The reader that does not want to get bothered with the details of this construction can just take Proposition 3.3 below for granted.

First we consider the category T_{Ass} which is the Lawvere theory of associative monoids. By definition $T_{\text{Ass}}^{\text{op}}$ is the full subcategory of the category of all assoiative monoids (in Set) which are free on a finite set S. Thus isomorphism classes are labeled by natural numbers (including 0) and the sets of maps are given by the sets of products of associative monoids. By definition T_{Ass} has finite products. An associative monoid object M in the ∞-category of spaces is then given by a functor

$$M\colon \quad T_{\text{Ass}} \to \mathcal{S}$$

which preserves finite products. This is not the definition in [Lur16] but equivalent to it, as follows from [Cra10] or [GGN15, Appendix B]. Now there is a functor $j : \tilde{\Lambda} \to T_{\text{Ass}}^{\text{op}}$ given by sending the object $[n]_{\tilde{\Lambda}}$ to the free associative monoid on n generators. A more precise way of defining j is as follows: let $\mathbb{T}_n^{\sim} \to \mathbb{T}_n$ be the inclusion of the discrete category on the set of objects of $\mathbb{T}_n$. This is natural in n, i.e. gives rise to a functor $\tilde{\Lambda} \to \text{Cat}^{\Delta^1}$. Then we form the cofibre $\mathbb{T}_n/\mathbb{T}_n^{\sim}$ of this map in the 2-category Cat of categories [3] which is canonically a pointed object in Cat and take the endomorphisms of the basepoint:

$$j([n]_{\tilde{\Lambda}}) := \text{End}_{\mathbb{T}_n/\mathbb{T}_n^{\sim}}(\text{pt}) \ .$$

This is a monoid and can easily be seen to be free on n generators. In fact the category $\mathbb{T}_n/\mathbb{T}_n^{\sim}$ has up to equivalence on object and thus is determined by this monoid. This construction of j makes the functoriality evident. The (epi)cyclic Bar construction of M is then defined to be the composition

$$\tilde{\Lambda}^{\text{op}} \xrightarrow{j^{\text{op}}} T_{\text{Ass}} \xrightarrow{M} \mathcal{S} \ .$$

Explicitly this means that the n-th level of this epicylic object is given by maps of associative monoids from the free monoid on n-generators to M, i.e. M^n. Thus one finds that the underlying simplicial object of this epicyclic object takes up to equivalence the form

$$\cdots \quad M \times M \times M \quad M \times M \longrightarrow M$$

which is the usual form of the cyclic Bar construction. The point is that due to the fact that we work in a Cartesian symmetric monoidal category the usual cyclic Bar construction extends to an epicyclic object.

Proposition 3.3. *If M is an associative monoid in the ∞-category $\mathcal{S}$ of spaces then the space* uTHH(BM) *for the associated ∞-category BM is equivalent to the geometric realization of the epicyclic Bar construction of M. This equivalence is compatible with the induced $\mathbb{T} \rtimes \mathbb{N}_{>0}$-actions.*

Proof. The cofibre sequence $(\mathbb{T}_n^{\sim})_+ \to (\mathbb{T}_n)_+ \to \mathbb{T}_n/\mathbb{T}_n^{\sim}$ in $(\text{Cat}_\infty)_*$ gives by mapping it to BM with the canonical basepoint rise to a fibre sequence of epicylic spaces

$$\text{Fun}_*(\mathbb{T}_n/\mathbb{T}_n^{\sim}, BM)^{\sim} \to \text{Fun}(\mathbb{T}_n, \mathcal{C})^{\sim} \to (BM^{\times})^n \tag{2}$$

where Fun_* denotes functors under the ∞-category pt $= \Delta^0$. Since $\mathbb{T}_n/\mathbb{T}_n^{\sim}$ is the free monoid on n-generators (this is how j was defined) and since pointed functors $BM' \to BM$ for monoids M and M' are equivalent to the space of maps of monoids $M \to M'$ it follows that the first epicylic space is equivalent, as an epicyclic space to the epicylic Bar construction of M as defined above. Therefore from the first

[3]For this cofibre it also does not matter if we consider Cat as a 1- or 2-category. In fact we could even define it to be the cofibre in ∞-categories which is what we will use soon.

map in (2) we get after realization a map from the realization of the epicyclic Bar construction to $\mathrm{uTHH}(\mathcal{C})$. This map is automatically $\mathbb{T} \rtimes \mathbb{N}_{>0}$-equivariant since it comes from a map of epicyclic objects.

Every map of spaces $X \to BG$ where G is the classifying space of a group is automatically of the form $Y_{hG} \to BG$ where Y is the fibre of the map $X \to BG$. In other words X is the homotopy quotient of a G-action on X. Similarly we see that the map of epicylic objects

$$\mathrm{Fun}_*(\mathbb{T}_n/\mathbb{T}_n^{\sim}, BM)^{\sim} \to \mathrm{Fun}(\mathbb{T}_n, BM)^{\sim}$$

is the homotopy quotient in epicyclic spaces by an action of the epicyclic group $G_\bullet : [n] \mapsto (M^\times)^n$. Since geometric realization is a colimit this implies that $\mathrm{uTHH}(BM)$ is the homotopy quotient of the geometric realization of the epicylic Bar construction by an action of the realization of $G_\bullet$. But the realization of $G_\bullet$ is contractible since $G_\bullet$ is contractible as a simplicial space which is easy to see (it for example admits an extra degeneracy). In fact this is the standard model for the contractible space EG. This concludes the proof. □

An immediate consequence of the definition of unstable cyclic homology the way we defined it is the following well-known result of Goodwillie-Jones (which is in the epicyclic version proven in [BFG94]).

Proposition 3.4. *Let X be a space (i.e. a Kan complex) considered as an ∞-category. Then we have an equivalence*

$$\mathrm{uTHH}(X) \simeq LX \ .$$

where $LX = \mathrm{Map}(\mathbb{T}, X)$ is the free loop space of X. This equivalence is compatible with the $\mathbb{T} \rtimes \mathbb{N}_{>0}$-actions on both sides, where on the free loop space $\mathbb{T} \rtimes \mathbb{N}_{>0}$ acts through its action on $\mathbb{T}$.

Proof. For X an ∞-groupoid we have that

$$\mathrm{Fun}(\mathrm{N}\mathbb{T}_n, X)^{\sim} \simeq \mathrm{Fun}\left(|\mathrm{N}\mathbb{T}_n|, X\right) \simeq \mathrm{Map}\left(\mathbb{T}, X\right)$$

and the underlying simplicial object for varying n is constant. This shows that as spaces without action we have the desired equivalence $\mathrm{uTHH}(X) \simeq LX$. As an epicyclic object we can describe

$$[n]_{\tilde{\Lambda}} \mapsto \mathrm{Fun}(\mathrm{N}\mathbb{T}_n, X)^{\sim} \simeq LX$$

by sending the cyclic operator on the n-th space to rotation by $\frac{2\pi}{n}$. The map $[kn]_{\tilde{\Lambda}} \to [n]_\Lambda$ maps to the n-fold covering of S^1. This is strictly true on the level of geometric realizations of $\mathbb{T}_n$ and therefore on the geometric realization and thus implies the claim. □

4. Coefficients and categories of endomorphisms

In this section we want to allow coefficients for uTHH similar to the fact that usual Hochschild homology allows for bimodules as coefficients. This section is of rather technical and combinatorial nature but will be essential for the later constructions and definitions. The impatient reader can try to skip this section and only jump back to it as needed.

The idea is to generalize the construction which takes an object $[n]_\Lambda$ and an ∞-category $\mathcal{C}$ and produces the n-th layer of the cyclic Bar construction $\mathrm{Fun}(\mathrm{N}\mathbb{T}_n, \mathcal{C})^{\sim}$

to allow for more general inputs. In fact we will define a 'twisted' variant that takes as input a functor $\mathcal{D} \to \mathrm{N}T_n$ and produces the space of sections

$$E(\mathcal{D} \to \mathrm{N}\mathbb{T}_n) := \mathrm{Fun}_{\mathrm{N}T_n}(\mathrm{N}\mathbb{T}_n, \mathcal{D})^{\sim}$$

so that for $\mathcal{D} = \mathcal{C} \times \mathrm{N}\mathbb{T}_n$ we get back the space of functors $\mathrm{Fun}(\mathrm{N}\mathbb{T}_n, \mathcal{C})^{\sim}$ as in the definition of $\mathrm{uTHH}(\mathcal{C})$. Here $\mathrm{Fun}_{\mathrm{N}\mathbb{T}_n}(\mathrm{N}\mathbb{T}_n, \mathcal{D})^{\sim}$ means the simplicial set of strict sections and we will assume that $\mathcal{D} \to \mathrm{N}\mathbb{T}_n$ is.a categorical fibration, so that this has the 'correct' homotopy type (i.e. equivalent to the space of homotopy coherent sections). In fact we will impose a more restrictive condiiton.

Construction 4.1. We say that a functor $X \to \mathrm{N}\mathbb{T}_n$ is a flat categorical fibration if it is a categorical fibration as well as a flat fibration[4] . We denote the full subcategory of $(\mathrm{Cat}_\infty)_{/\mathrm{N}\mathbb{T}_n}$ consisting of the flat categorical fibrations by $(\mathrm{Cat}_\infty^\flat)_{/\mathrm{N}\mathbb{T}_n}$. The pullback of a flat categorical fibration is again a flat categorical fibration and thus we have a functor

$$\chi\colon \mathrm{N}\Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty \qquad [n]_\Lambda \mapsto (\mathrm{Cat}_{\infty/\mathrm{N}\mathbb{T}_n})^\flat$$

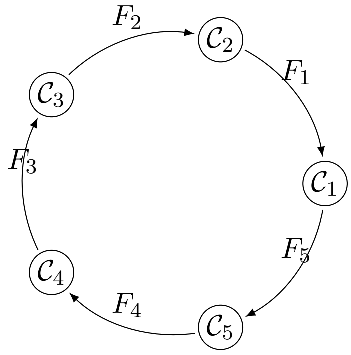
obtained as straightening of the Cartesian fibration given as $(\mathrm{Cat}_{\infty/\Lambda})^\flat \to \Lambda$ where $(\mathrm{Cat}_{\infty/\Lambda})^\flat \subseteq \mathrm{Cat}_{\infty/\Lambda}$ is the full subcategory consisting of flat fibrations. An altenative way of construction this functor is by consructing it as a pseduo-functor using the approach taken in [GHN15, Appendix A]. We leave the details to the reader.

Definition 4.2. *We let $\Lambda^{\mathrm{Cat}} \to \mathrm{N}\Lambda^{\mathrm{op}}$ be the total space of the coCartesian fibration classified by the functor χ of Construction 4.1. We refer to object of Λ^{Cat} as labelled cyclic graphs (labelled by ∞-categories and bimodules).*

Remark 4.3. To understand how Definition 4.2 implements 'cyclic graphs' we note that a flat categorical fibration $p : \mathcal{X} \to \Delta^1$ (over Δ^1 this is equivalently to p being an inner fibration and even to $\mathcal{X}$ being an ∞-category) is equivalent given by a colimit preserving functor $\mathcal{P}(\mathcal{X}_1) \to \mathcal{P}(\mathcal{X}_0)$ where $\mathcal{X}_i = p^{-1}(i)$.[5] The flatness condition for a fibration $\mathcal{X} \to S$ ensures that the composition in S corresponds to the composition of functors on presheaf categories. One should think of an object $\mathcal{C} \to \mathrm{N}\mathbb{T}_n$ in Λ^{Cat} as giving a cyclic graph labelled with ∞-categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and colimit preserving functors $F_i : \mathcal{P}(\mathcal{C}_{i+1}) \to \mathrm{Ind}\mathcal{P}(\mathcal{C}_i)$ for every i. We shall abbreviate such an object as $(F_1, \ldots, F_n)$ leaving the ∞-categories implicit or even as $\vec{F}$ and depict it as follows (for $n = 5$):

[4]This means that for every 2-simplex $\Delta^2 \to S$ the induced map $X \times_S \Lambda^2_1 \to X \times_S \Delta^2$ is a categorical equivalence., see [Lur16]

[5]Here $\mathcal{P}(\mathcal{X}_i) = \mathrm{Fun}(\mathcal{X}_i^{\mathrm{op}}, \mathcal{S})$ is the presheaf category. The datum of a colimit preserving functor $\mathcal{P}(\mathcal{X}_1) \to \mathcal{P}(\mathcal{X}_0)$ is equivalent to a functor $\mathcal{X}_1 \times \mathcal{X}_0^{\mathrm{op}} \to \mathcal{S}$, i.e. a profunctor (a.k.a. bimodule or correspondence).



Here the functors are really profunctors but we draw them as ordinary arrows. The morphisms in Λ^{Cat} are generated by the following basic morphisms: composing adjacent functors, inserting identities, rotations and (lax) maps of diagrams of fixed shape. Lax means to for a given shape there are 2-cells allowed. For example if we have two endoprofunctors $F : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ and $G : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D})$ considered as an object in $(\Lambda^{\mathrm{Cat}})_{[1]_\Lambda}$ then a morphism in $(\Lambda^{\mathrm{Cat}})_{[1]_\Lambda}$ corresponds to a pair of a functor $\phi : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta : \phi_! \circ F \to G \circ \phi_!$ filling the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{C}) & \xrightarrow{F} & \mathcal{P}(\mathcal{C}) \\ \downarrow{\scriptstyle \phi_!} & \swarrow\!\!\!\!/\,\eta & \downarrow{\scriptstyle \Phi_!} \\ \mathcal{P}(\mathcal{D}) & \xrightarrow{G} & \mathcal{P}(\mathcal{D}) \end{array} .$$

In particular for $\mathcal{C} = \mathcal{D}$ and $\phi = \mathrm{id}$ there are still morphisms of profunctors (aka bimodules) built into Λ^{Cat}. This will become very important in Section 5.

Remark 4.4. Note that one can use the results of [BGN18] to give a very explicit description of Λ^{Cat} since the classifying functor $\chi\colon \mathrm{N}\Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is itself obtained by straightening (but with the different variance). This will give a description of Λ^{Cat} in terms of a certain ∞-category of spans in $\mathrm{Cat}_\infty^{\Delta^1}$: objects are still flat categorical fibrations $\mathcal{X} \to \mathrm{N}\mathbb{T}_n$ but a morphism from $\mathcal{Z} \to \mathrm{N}\mathbb{T}_n$ to $\mathcal{X} \to \mathrm{N}\mathbb{T}_n$ is given by a span

$$\begin{array}{ccccc} \mathcal{X} & \longleftarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathrm{N}\mathbb{T}_i & \xleftarrow{\sim} & \mathrm{N}\mathbb{T}_j & \longrightarrow & \mathrm{N}\mathbb{T}_k \end{array}$$

where the left lower map is induced from an isomorphism in Λ and the right square is a pullback square in Cat_∞ (or even a strict pullback since the right vertical morphism is a categorical fibration). We shall not need this explicit description here and thus go with the abstract definition. But if one uses this explicit model then there is a more direct proof of the next result.

Proposition 4.5. *There is a functor*

$$\mathrm{End} : \Lambda^{\mathrm{Cat}} \to \mathrm{Cat}_\infty$$

such that objectwise we have $\mathrm{End}(X \to \mathrm{N}\mathbb{T}_n) \simeq \mathrm{Fun}_{\mathrm{N}\mathbb{T}_n}(\mathrm{N}\mathbb{T}_n, X)$.

Proof. We first construct a functor

$$\chi_* : \mathrm{N}\Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty \qquad [n]_\Lambda \mapsto (\mathrm{Cat}_{\infty/\mathrm{N}\mathbb{T}_n})^\flat_\star$$

where $(\mathrm{Cat}_{\infty/\Lambda})^\flat_\star$ denotes the ∞-category of flat fibrations equipped with a section. More precisely the ∞-category $(\mathrm{Cat}_{\infty/\Lambda})^\flat_\star$ is constructed as follows:

Note that the ∞-category $(\mathrm{Cat}_{\infty/\Lambda})^\flat_\star$ is not just the ∞-category of pointed objects in the ∞-category $(\mathrm{Cat}_{\infty/\Lambda})^\flat$. Its rather the 'lax slice' under the point: objects are pointed objects but morphisms only need to preserve the point just up to a non-necessarily invertible 2-cell. In fact there is a non-full inclusion $(\mathrm{Cat}_{\infty/\Lambda})^\flat_* \subseteq (\mathrm{Cat}_{\infty/\Lambda})^\flat_\star$ from pointed objects into our ∞-category.

This functor is obtained from the functor χ from Construction 4.1 by postcomposing with the functor that sends an ∞-category $\mathcal{C}$ which has a terminal object to the ∞-category $\mathcal{C}_*$ of pointed objects, which is the slices under the terminal object. To apply this construction one has to invoke that $(\mathrm{Cat}_{\infty/\mathrm{N}\mathbb{T}_n})$ has a terminal object (namely the identity) and that this terminal object is preserved by pullback along any map $\mathrm{N}\mathbb{T}_n \to \mathrm{N}\mathbb{T}_m$. Now by construction there is.a natural forgetful transformation

$$\chi_* \to \chi$$

and for each object $[n]_\Lambda$ of Λ the functor

$$\chi_*([n]_\Lambda) = (\mathrm{Cat}_{\infty/\mathrm{N}\mathbb{T}_n})^\flat_* \longrightarrow (\mathrm{Cat}_{\infty/\mathrm{N}\mathbb{T}_n})^\flat = \chi([n]_\Lambda)$$

is itself a coCartesian fibration[6]. To see this we note that more generally for each ∞-category $\mathcal{C}$ with a terminal object pt the forgetful functor $\mathcal{C}_* := \mathcal{C}_{\mathrm{pt}/} \to \mathcal{C}$ is coCartesian. We now invoke the dual of Proposition 9.6 of [GHN15] to deduce that the functor from the total space $\mathcal{X}$ of the coCartertisian fibration $\mathcal{X} \to \mathrm{N}\Lambda^{\mathrm{op}}$ classified by χ_* to the ∞-category Λ^{Cat} (which is the total space of the coCartesian fibration classified by χ) is itself a coCartesian fibration. Thus it is classified by a functor $\mathrm{End} : \Lambda^{\mathrm{Cat}} \to \mathrm{Cat}_\infty$ with the desired properties. □

Example 4.6. We have the object $\mathcal{C} \times \mathrm{N}\mathbb{T}_1 \to \mathrm{N}\mathbb{T}_1$ of Λ^{Cat} which we will abbreviate as $\mathcal{C}$ or more in line with the notation of Remark 4.3 as $(\mathrm{id}_{\mathcal{P}(C)})$. We find that $\mathrm{End}(\mathcal{C}) \simeq \mathrm{Fun}(\mathrm{N}\mathbb{T}_1, \mathcal{C})$ is the ∞-category whose objects are given by pairs consisting of an object $c \in \mathcal{C}$ together with an endomorphism $c \to c$. This is the most important case for us. More generally for every n we have an equivalence $\mathrm{End}(\mathcal{C} \times \mathrm{N}\mathbb{T}_n \mathrm{N}\mathbb{T}_n) \simeq \mathrm{Fun}(\mathrm{N}T_n, \mathcal{C})$.

For a general cyclic graph of stable ∞-categories $(F_1, \ldots, F_n)$ the ∞-category $\mathrm{End}(F_1, \ldots, F_n)$ has as objects sequences of objects $(c_i \in \mathcal{C}_i)$ together with morphisms $c_i \to F_i(c_{i+1})$.

Lemma 4.7. *For an object $\mathcal{C} \to \mathrm{N}\mathbb{T}_n$ corresponding to the list of functors $(F_1, \ldots, F_n)$ there is an equivalence*

$$\mathrm{End}(F_1, \ldots, F_n)^\sim \simeq \operatorname*{colim}_{c_i \in \mathcal{C}_i^\sim} \prod_{i=1}^n \mathrm{Map}_{\mathcal{P}(\mathcal{C}_i)}(c_i, F_i c_{i+1})$$

Proof. We consider the functor $\mathrm{N}\mathbb{T}_n^\sim \to \mathrm{N}\mathbb{T}_n$. Here $\mathbb{T}_n^\sim$ is is a discrete category on n objects. Then we get an induced functor

$$\mathrm{Fun}_{\mathrm{N}\mathbb{T}_n}(\mathrm{N}\mathbb{T}_n, \mathcal{C}) \to \mathrm{Fun}_{\mathrm{N}\mathbb{T}_n}(\mathrm{N}\mathbb{T}_n^\sim, \mathcal{C}) \simeq \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \ .$$

[6]Here we use coCartesian in the invariant way, meaning that any replacement by an inner fibration is coCartesian in the sense of [Lur09]

The induced map of spaces after taking groupoid cores take the form:

$$\mathrm{Fun}_{\mathrm{N}\mathbb{T}_n}(\mathrm{N}\mathbb{T}_n, \mathcal{C})^{\sim} \to \mathcal{C}_1^{\sim} \times \ldots \times \mathcal{C}_n^{\sim} \ .$$

Every map of spaces with target S is the colimit of a classifying functor $S \to \mathcal{S}$. Thus in our case get a functor $\mathcal{C}_1^{\sim} \times \ldots \times \mathcal{C}_n^{\sim} \to \mathcal{S}$. This functor is pointwise of the form

$$(c_1, \ldots, c_n) \mapsto \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{C}}(c_i, c_{i+1}) \ .$$

Finally we have $\mathrm{Map}_{\mathcal{C}}(c_i, c_{i+1}) \simeq \mathrm{Map}_{\mathcal{P}(\mathcal{C}_i)}(c_i, F_i c_{i+1})$ by construction of the functors F_i. □

Finally we can use the ∞-category Λ^{Cat} to give a definition of $\mathrm{uTHH}(\mathcal{C}, F)$ for an ∞-category $\mathcal{C}$ together with a colimit preserving functor $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$, i.e. a flat fibration over $\mathrm{N}\mathbb{T}_1$, as follows: we claim that in this situation there is a simplicial object

$$\mathrm{N}\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{Cat}}$$

which is informally given as

$$\ldots (F, \mathrm{id}, \mathrm{id}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (F, \mathrm{id}) \substack{\longrightarrow\\\longrightarrow} F$$

and will be constructed more carefully in Construction 4.8 below. Then we define $\mathrm{uTHH}(\mathcal{C}, F)$ as the colimit of the functor $\mathrm{End}(-)^{\sim}$ applied to this diagram. More generally for every object $\vec{F} = (F_1, \ldots, F_n)$ in Λ^{Cat} we get a simplicial object

$$\ldots (F_1, \mathrm{id}, \mathrm{id}, F_2, \mathrm{id}, \mathrm{id}, \ldots, F_n, \mathrm{id}, \mathrm{id}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (F_1, \mathrm{id}, F_2, \mathrm{id}, \ldots, F_n, \mathrm{id}) \substack{\longrightarrow\\\longrightarrow} (F_1, \ldots, F_n) \ .$$

Note that these objects are neither cyclic, paracyclic or epicyclic but just simplicial in general. Only if we input a list merely consisting of identities we do get the additional structure. We will make this precise below.

Construction 4.8. There is a functor

$$r : \Lambda^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$$

given by sending $P \in \Lambda$ and $S \in \Delta$ to $P \times S$ equipped with the lexicographic ordering (first compare the entries in P and then those in S) and the $\mathbb{Z}$-action in the first factor. We want to construct a diagram of functors

$$\begin{array}{ccc} \Lambda^{\mathrm{Cat}} \times \mathrm{N}\Delta^{\mathrm{op}} & \xrightarrow{R} & \Lambda^{\mathrm{Cat}} \\ \downarrow & & \downarrow \\ \mathrm{N}\Lambda^{\mathrm{op}} \times \mathrm{N}\Delta^{\mathrm{op}} & \xrightarrow{r} & \mathrm{N}\Lambda^{\mathrm{op}} \ . \end{array}$$

where S sends coCartesian lifts to coCartesian lifts. This is by definition the same as a map of coCartesian fibrations

$$\begin{array}{ccc} \Lambda^{\mathrm{Cat}} \times \mathrm{N}\Delta^{\mathrm{op}} & \longrightarrow & r^* \Lambda^{\mathrm{Cat}} \\ & \searrow \quad \swarrow & \\ & \mathrm{N}\Lambda^{\mathrm{op}} \times \mathrm{N}\Delta^{\mathrm{op}} & \end{array}$$

The two coCartesian fibrations are classified by the functors

$$\chi_1 \colon \mathrm{N}\Lambda^{\mathrm{op}} \times \mathrm{N}\Delta^{\mathrm{op}} \xrightarrow{\mathrm{pr}_{\Lambda^{\mathrm{op}}}} \mathrm{N}\Lambda^{\mathrm{op}} \xrightarrow{\chi} \mathrm{Cat}_{\infty}$$

and

$$\chi_2 \colon \mathrm{N}\Lambda^{\mathrm{op}} \times \mathrm{N}\Delta^{\mathrm{op}} \xrightarrow{r} \mathrm{N}\Lambda^{\mathrm{op}} \xrightarrow{\chi} \mathrm{Cat}_\infty \ .$$

We observe that there is a natural transformation of functors $\mathrm{pr}_{\Lambda^{\mathrm{op}}} \to r$ given by the projection $P \times S \to P$. This induces a natural transformation $\chi_1 \to \chi_2$ and then a functor as desired on coCartesian fibrations. Unfolding the definitions we see that

$$R \colon \ \Lambda^{\mathrm{Cat}} \times \mathrm{N}\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{Cat}}$$

sends a pair $(S, \mathcal{D} \to \mathrm{N}\mathbb{T}_P)$ to the flat fibration $\mathrm{pr}^*\mathcal{D} \to \mathrm{N}\mathbb{T}_{P\times S}$ where pr is induced by the projection map $P \times S \to P$.

Definition 4.9. *We get a functor* uTHH : $\Lambda^{\mathrm{Cat}} \to \mathcal{S}$ *as the composite*

$$\Lambda^{\mathrm{Cat}} \xrightarrow{R} \left(\Lambda^{\mathrm{Cat}}\right)^{\mathrm{N}\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{End}} (\mathrm{Cat}_\infty)^{\mathrm{N}\Delta^{\mathrm{op}}} \xrightarrow{(-)^\sim} \mathcal{S}^{\mathrm{N}\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{colim}} \mathcal{S}$$

By construction the composition $\mathrm{Cat}_\infty^{\mathrm{st}} \to \Lambda^{\mathrm{Cat}} \to \mathcal{S}$ is equivalent to the definition of uTHH given above.

We end this section by giving the twisted version of the structure discussed at the end of the first section.

Example 4.10. For every functor $T : \Lambda^{\mathrm{Cat}} \to \mathcal{S}$ we can form a new functor T^{hC_p} : $\Lambda^{\mathrm{Cat}} \to \mathcal{S}$ informally defined as follows:

$$T^{hC_p}(\vec{F}) = T\big(\underbrace{\vec{F}, \vec{F}, \dots, \vec{F}}_{p \text{ times}}\big)^{hC_p}$$

The functoriality of this construction is given as follows: first we claim that the morphism

$$sd_p : \Lambda_p \to \Lambda \qquad [n]_{\Lambda_p} \to [n]_\Lambda$$

lifts to a functor $\Lambda_p^{\mathrm{st}} \to \Lambda^{\mathrm{st}}$ given by sending $X \to \mathbb{T}_n$ to the flat stable fibration $w^*X \to (pn)$ where $w : (pn) \to (n)$ where w is the map that wraps around p-times. In fact this functor moreover lifts to a BC_p-equivariant functor

$$\Lambda_p^{\mathrm{st}} \to (\Lambda^{\mathrm{st}})^{BC_p}$$

so that we get an induced functor $T^{hC_p} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ as in Section 3.

Example 4.11. There is a natural transformation $E \to E^{hC_p}$ sending a sequence of morphisms $(\varphi_1, \dots, \varphi_p)$ to the p-fold iterate. Formally there is an equivalence of the ∞-categories of sections

$$\mathrm{End}(X \to \mathrm{N}\mathbb{T}_n) \simeq \mathrm{End}(w^*X \to \mathrm{N}\mathbb{T}_{pn})^{hC_p}$$

which gives the transformation.

5. Stable ∞-categories and stable theories

In this section we want to define a variant of uTHH as discussed in the previous sections, called THH($\mathcal{C}$) where we replace $\mathcal{C}$ by a stable ∞-category and such that THH($\mathcal{C}$) is a spectrum in contrast to the space uTHH($\mathcal{C}$). The idea is to define to define THH($\mathcal{C}$) as the geometric realization of a cyclic object which is informally given by

$$[n]_\Lambda \mapsto \operatorname*{colim}_{c_1,\dots,c_n \in \mathcal{C}^\sim} \bigotimes_{i=1}^{n} \mathrm{map}_{\mathcal{C}}(x_i, F_i x_{i+1}) \tag{3}$$

where $\mathrm{map}_{\mathcal{C}}(-,-)$ denotes the mapping spectum in $\mathcal{C}$. This generalizes the description in Remark 3.2.

We define an ∞-category of 'labelled cyclic graphs' Λ^{st} analogously to the ∞-category Λ^{Cat} from above. Again, up to equivaelence, an object in Λ^{st} is given by a cyclic graph labelled with stable ∞-categories $\mathcal{C}_1, \dots, \mathcal{C}_n$ and colimit preserving functors $F_i : \mathrm{Ind}\mathcal{C}_i \to \mathrm{Ind}\mathcal{C}_{i+1}$ for every i. As before we will abbreviate such an object as $(F_1, \dots, F_n)$ leaving the stable ∞-categories implicit or even as $\vec{F}$. To make the definition precise we need the following notion.

Definition 5.1. *Let $X \to S$ be a categorical fibration of simplcial sets. We say that that it is a flat stable fibration if it is a flat categorical fibration, every fibre X_s for $s \in S$ is a stable ∞-category and for every edge $s \to s'$ in S the induced map $X_s^{\mathrm{op}} \times X_{s'} \to \mathcal{S}$ is excisive in every variable separately* [7]. *A functor $X \to X'$ of flat stable fibrations over S is a functor of fibrations over S which is fibrewise exact. We denote the ∞-category of flat stable fibrations over S (considered as a subcategory of the slice category) by $\mathrm{Stab}^{\flat}_{/S}$.*

As in Construction 4.1 there is a functor

$$\chi : \Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty \qquad [n]_\Lambda \mapsto \mathrm{Stab}^{\flat}_{/\mathrm{N}\mathbb{T}_n}$$

and we let Λ^{st} be the coCartesian fibration over Λ^{op} classifying it. We refer to objects of Λ^{st} as cyclic graphs of stable ∞-categories.

Note that the coCartesian morphisms of Λ^{st} over Λ^{op} are compositions of contraction, insertion and rotation morphisms. In the following we will study functors $T\colon \Lambda^{\mathrm{stab}} \to \mathrm{Sp}$. In fact most of what we say makes sense for functors with target an arbitrary ∞-category. In particular we will realize topological Hochschild homology and cyclic K-theory in this paper as such functors. Let us start by an example.

Example 5.2. There is a functor $\Lambda^{\mathrm{st}} \to \Lambda^{\mathrm{Cat}}$ given by forgetting that a fibration is stable. Then we get a resulting functor

$$E : \Lambda^{\mathrm{st}} \to \Lambda^{\mathrm{Cat}} \xrightarrow{\mathrm{End}} \mathrm{Cat}_\infty \xrightarrow{(-)^\sim} \mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}\ .$$

which will be key for what will follow. We also get a natural transformation $E \to E^{hC_p}$.

Construction 5.3. For every sequence of stable ∞-categories $(\mathcal{C}_1, \dots, \mathcal{C}_n)$ we construct a functor

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_i, \mathrm{Ind}\mathcal{C}_{i+1}) \to \Lambda^{\mathrm{st}}$$

where $\mathrm{Fun}^{\mathrm{L}}$ denotes colimit preserving (equivalently left adjoint) functors. It sends the sequence $(F_1, \dots, F_n)$ to the object denoted in the same way in Λ^{st}.

Definition 5.4. *Let $T\colon\ \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ be a functor.*

- *T is called* reduced *if for every sequence of stable ∞-categories $\mathcal{C}_1, \dots, \mathcal{C}_n$ the restriction of T to a functor*

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_i, \mathrm{Ind}\mathcal{C}_{i+1}) \to \mathrm{Sp}$$

[7] Such a functor is equivalently given by a colimit preserving functor $\mathrm{Ind}X_{s'} \to \mathrm{Ind}X_s$

is reduced in every variable separately, i.e. sends the zero functor to a zero object in Sp *(as usual i is taken mod n)*

- *T is called* stable *if for every sequence of stable ∞-categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the restriction of T to a functor*

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_i, \mathrm{Ind}\mathcal{C}_{i+1}) \to \mathrm{Sp}$$

is exact in every variable separately, i.e. sends pushouts in $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Ind}C_i, \mathrm{Ind}C_{i+1})$ *to pullbacks in* Sp.

Proposition 5.5. *The inclusions*

$$\mathrm{Fun}^{\mathrm{stab}}(\Lambda^{\mathrm{st}}, \mathrm{Sp}) \subseteq \mathrm{Fun}^{\mathrm{red}}(\Lambda^{\mathrm{stab}}, \mathrm{Sp}) \subseteq \mathrm{Fun}(\Lambda^{\mathrm{stab}}, \mathrm{Sp})$$

admit left adjoints $T \mapsto T^{\mathrm{stab}}$ and $T \mapsto T^{\mathrm{ptd}}$ such that

$$T^{\mathrm{st}}(F_1, \ldots, F_n) \simeq \varinjlim_k \Omega^{nk} T\big(\Sigma^k F_1, \ldots, \Sigma^k F_n\big) \ .$$

and $T^{\mathrm{red}}(F_1, \ldots, F_n)$ is given by the total cofibre of the n-cube

$$P\{1, \ldots, n\} \to \mathrm{Sp} \qquad S \subseteq \{1, \ldots, n\} \mapsto T(F_1', \ldots, F_n')$$

where F_i' is given by F_i for $i \in S$ and by the zero functor otherwise.

Proof. The idea is to use Theorem A.1 in the Appendix for the coCartesian fibration $\Lambda^{\mathrm{st}} \to \Lambda^{\mathrm{op}}$. To apply this we have to check in particular that for every fibre ${\Lambda^{\mathrm{st}}}_{[n]}$ there is a localization of $\mathrm{Fun}\big((\Lambda^{\mathrm{st}})_{[n]}, \mathrm{Sp}\big)$ whose local objects are the 'stable' functors. To see this we contemplate the fibre of Λ^{st} over $[n]_\Lambda \in \Lambda$. On this fibre there is an endofunctor which sends $(F_1, \ldots, F_n)$ to $(\Sigma F_1, \ldots, \Sigma F_n)$. Using this endofunctor we can define the localization on the fibre and then use the result of the appendix to extend it.

TODO

□

We will abusively denote the composite left adjoint

$$\mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{stab}}(\Lambda^{\mathrm{stab}}, \mathrm{Sp})$$

given by first making the functor reduced and then stable also by $(-)^{\mathrm{stab}}$. The results above show that the restriction of the transformation $T \to T^{\mathrm{st}}$ to the subcategory

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_i, \mathrm{Ind}\mathcal{C}_{i+1}) \subseteq \Lambda^{\mathrm{st}}$$

exhibit $T^{\mathrm{st}} |_{\prod_{i=1}^n \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_i, \mathrm{Ind}\mathcal{C}_{i+1})}$ as the exact (i.e. 1-excisive and reduced) approximation of $T |_{\prod_{i=1}^n \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_i, \mathrm{Ind}\mathcal{C}_{i+1})}$.

Now we will try to understand the stable approximation E^{st} to the functor $E : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ as discussed above. To this end we will need the following result.

Lemma 5.6. *Let $\mathcal{C}$ be a stable ∞-category with an object $c \in C$. Then the exact approximation of the functor*

$$\Sigma^\infty_+ \mathrm{Map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Sp}$$

is given by the mapping spectrum functor

$$\mathrm{map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Sp} \ .$$

Proof. Let us first recall that the mapping spectrum functor $\mathcal{C} \to \mathrm{Sp}$ can be characterised (and defined) as follows: it is the unique product preserving functor $F : \mathcal{C} \to \mathrm{Sp}$ with the property $\Omega^\infty F \simeq \mathrm{Map}_{\mathcal{C}}(c, -)$. This works since the functor

$$(\Omega^\infty)_* : \mathrm{Fun}^{\mathrm{LEx}}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{LEx}}(\mathcal{C}, \mathcal{S})$$

is an equivalence of ∞-categories, where Fun^{LEx} are left exact, i.e. finite limit preserving functors. Adjoint to the equivalence $\mathrm{Map}_{\mathcal{C}}(c, -) \simeq \Omega^\infty F$ we get a map $p : \Sigma^\infty_+ \mathrm{Map}_{\mathcal{C}}(c, -) \to F$ and we claim that this maps exhibits F as the exact approximation. It suffices to verify the universal property. Thus let $G : \mathcal{C} \to \mathrm{Sp}$ be an exact (equivalently right exact0 functor. Then we get the following commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(F, G) & \xrightarrow[\simeq]{\Omega^\infty_*} & \mathrm{Map}(\Omega^\infty F, \Omega^\infty G) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Map}(\Sigma^\infty_+ \mathrm{Map}_{\mathcal{C}}(c, -), G) & \xrightarrow{\simeq} & \mathrm{Map}(\mathrm{Map}_{\mathcal{C}}(c, -), \Omega^\infty G) \end{array}$$

which shows that the left vertical map is an equivalence. □

Proposition 5.7. *The exact approximation to the functor* $E : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *of Example 5.2 is pointwise given by*

$$E^{\mathrm{st}}(F_1, \ldots, F_n) \simeq \operatorname*{colim}_{c_1, \ldots, c_n \in \mathcal{C}^\sim} \bigotimes_{i=1}^{n} \mathrm{map}_{\mathcal{C}}(x_i, F_i x_{i+1}) \ .$$

Proof. By what we have shown above the functor $E^{\mathrm{st}}(F_1, \ldots, F_n)$ as a functor in $F_1, \ldots, F_n$, i.e. from

$$\prod_{i=1}^{n} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_{i+1}, \mathrm{Ind}\mathcal{C}_i) \to \mathrm{Sp}$$

is the exact (in the multivariabe sense) approximation to $E(F_1, \ldots, F_n)$. But for the latter we have the formula

$$\operatorname*{colim}_{c_1, \ldots, c_n \in \mathcal{C}^\sim} \bigotimes_{i=1}^{n} \Sigma^\infty_+ \mathrm{Map}_{\mathcal{C}}(x_i, F_i x_{i+1})$$

so that we have to compute the exact approximation (in F_i) to this one. Since colimit is exact and the tensor product is exact in each variable this comes down to compute the exact approximation to the functor

$$\Sigma^\infty_+ \mathrm{Map}_{\mathcal{C}}(x_i, -x_{i+1}) : \quad \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}\mathcal{C}_{i+1}, \mathrm{Ind}\mathcal{C}_i) \to \mathrm{Sp}$$

for fixed x_i and x_{i+1}. This is then essentially by the last Lemma given by the mapping spectrum so that he full claim follows. □

Remark 5.8. At this point wie note that every other construction of $E^{\mathrm{st}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ would work equally well for what follows. In particular if one prefers to work with strictly point-set enriched spectral categories on can give a much more direct construction of E^{st} (the author has not thought through all the details and how to get the full functoriality, it would be interesting to see this spelled out).

6. Topological Hochschild homology and trace theories

In the last section we have managed to product a functor

$$E^{\mathrm{st}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$$

that gives the 'layers' of the cyclic Bar construction. Thus we can now define topological Hochschild homology through the usual cyclic Bar construction as

$$\mathrm{THH}(\mathcal{C}) = \Big| \ldots E^{\mathrm{st}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \substack{\longrightarrow\\[-1ex] \longrightarrow\\[-1ex] \longrightarrow} E^{\mathrm{st}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \rightrightarrows E^{\mathrm{st}}(\mathrm{id}_{\mathcal{C}}) \Big| \ .$$

In fact we can define THH now more generally.

Definition 6.1. *We more generally define* $\mathrm{THH}(F_1, \ldots, F_n)$ *as the realization of the simplicial object*

$$\ldots \substack{\longrightarrow\\[-1ex] \longrightarrow\\[-1ex] \longrightarrow} E^{\mathrm{st}}(F_1, \mathrm{id}, F_2, \mathrm{id}, \ldots, F_n, \mathrm{id}) \rightrightarrows E^{\mathrm{st}}(F_1, \ldots, F_n)$$

where the simplicial object is obtained from the simplicial objects in Λ^{st} *obtained invoking the stable invariant of the functor* R *discussed in Construction 4.8 . This gives us a functor*

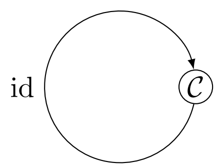
$$\mathrm{THH}\colon \Lambda^{\mathrm{st}} \to \mathrm{Sp}$$

which comes with a transformation $E^{\mathrm{st}} \to \mathrm{THH}$.

In particular we have a functor $\mathrm{Cat}_\infty^{\mathrm{stab}} \to \mathrm{Stab}^\flat_{/\mathrm{N}\mathbb{T}_1} \to \Lambda^{\mathrm{st}}$ sending $\mathcal{C}$ to

$$(\mathrm{id}_{\mathcal{P}(\mathcal{C})}) = (\mathcal{C} \times \mathrm{N}\mathbb{T}_1 \to \mathrm{N}\mathbb{T}_1)$$

i.e. the cyclic graph



The composition with THH as defined above then gives THH for stable ∞-categories.

Definition 6.2. *A functor* $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is called a* trace theory *if it sends coCartesian morphisms in* Λ^{st} *to equivalences in* Sp. *It is called a stable trace theory, if it additionally is stable (see Definition 5.4).*

For a trace theory T we have equivalences $T(F \circ G) \simeq T(F, G) \simeq T(G, F) \simeq T(G \circ F)$. Thus it behaves like the usual cyclic invariance of the trace of an endomorphism. This is the reason for the naming. In fact, for a trace theory $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ the values in $T(F)$ for a single functor already determine all values, since we have

$$T(F_1, \ldots, F_n) \simeq T(F_1 \circ \ldots \circ F_n) \ .$$

The main point of the notion of trace theory is that the combinatorics of Λ^{st} encodes some homotopy coherent way of expressing the 'cyclic invariance'. Also note that every coCartesian morphism is a composition of rotation, contraction and insertion morphisms (since this is true in Λ). The rotations are equivalence in Λ^{st} and the insertions are one sided inverses to contractions. Therefore to check that something is a trace theory it suffices to check that contraction of two adjacent morphisms induces an equivalence of spectra.

Theorem 6.3. *The functor* THH : $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is a stable trace theory. Moreover the natural transformation* $E \to \mathrm{THH}$ *exhibits* THH : $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *as the universal stable trace theory under* E*.*

Proof. First stability for the functor THH is clear by construction, since the cyclic Bar construction commutes with cofibre sequences.

First we will prove that THH is a trace theory. To this end we shall prove that the morphism $\mathrm{THH}(F, G) \to \mathrm{THH}(F \circ G)$ for two functors

$$F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D} \qquad \text{and} \qquad G : \mathrm{Ind}\mathcal{D} \to \mathrm{Ind}\mathcal{C}$$

is an equivalence. The proof for two adjacent morphisms in a list then works exactly the same (since the other morphisms will not be relevant). This then proves the claim.

For the composition of arbitrary produnctors F and G and and objects $d, d' \in \mathcal{D}$ we can compute $\mathrm{Map}_{\mathrm{Ind}(\mathcal{D})}(d, F(G(c))$ as the geometric realization of the simplicial object whose first three layers are

$$\operatorname*{colim}_{c_0,c_1,c_2 \in C^\sim} \Big(\mathrm{map}_{\mathrm{Ind}(\mathcal{D})}(d, Fc_0) \otimes \mathrm{map}(c_0, c_1) \otimes \mathrm{map}(c_1, c_2) \otimes \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(c_2, Gd')\Big)$$

$$\Rrightarrow \mathrm{colim}_{c_0,c_1 \in C^\sim} \Big(\mathrm{map}_{\mathrm{Ind}(\mathcal{D})}(d, Fc_0) \otimes \mathrm{map}(c_0, c_1) \otimes \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(c_1, Gd')\Big)$$

$$\rightrightarrows \mathrm{colim}_{c_0 \in C^\sim} \Big(\mathrm{map}_{\mathrm{Ind}(\mathcal{D})}(d, Fc_0) \otimes \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(c_0, Gd')\Big)$$

and which continues in the obvious way (this is essentially the usual Bar resolution of the tensor product of bimodules). We now plug this formula into the definition of $\mathrm{THH}(FG)$ in combination with the formula for the layers of the cyclic Bar construction (see Lemma 5.7). This way we obtain a bisimplicial spectrum. The diagonal of this bismplicial spectrum is exactly the cyclic Bar construction $\mathrm{THH}(F, G)$. Note that we have built in the correct functoriality already (we are just verifying a property of THH) so that the we have the maps a priori and do only need to verify a pointwise equivalence. It follows that the canononical map $\mathrm{THH}(FG) \to \mathrm{THH}(F, G)$ is an equivalence, since the colimit over $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is equivalent to the diagonal colimit by siftedness of Δ.

Now we want to verify that the transformation $E \to \mathrm{THH}$ exhibits THH as the initial stable trace theory under E. This transformation factors by construction as

$$E \to E^{\mathrm{st}} \to \mathrm{THH} \; .$$

We will show that $E^{\mathrm{st}} \to \mathrm{THH}$ exbhibits THH as the initial trace theory under E^{st}. This is enough, since THH is also stable and since $E \to E^{\mathrm{st}}$ is the initial stable theory. To do this we apply the criterion given in Proposition B.1. Taking the cyclic Bar construction defines an endofunctor $L : \mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathrm{Sp}) \to \mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathrm{Sp})$ with a transformation $\mathrm{id} \to L$. For objects in the full subcategory $\mathrm{Fun}^{\mathrm{trace}}(\Lambda^{\mathrm{st}}, \mathrm{Sp})$ the map $T \to LT$ is an equivalence. Moreover there is an autoequivalence $L^2T \to L^2T$ that flips the two directions of the bisimplicial functor that is realized to give L^2T. This flip translates between the two maps $LT \to L^2T$ so that they are equivalent in the arrow category and Proposition B.1 applies to give the claim. □

Remark 6.4. The definition of THH can also be given as an actual categorical trace as follows: we consider the category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of presentable stable ∞-categories and left adjoint functors. It admits a (closed) symmetric monoidal structure constructed by

Lurie. The tensor product corepresents functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that preserve colimits in both variables separately. It turns out that every compactly generated ∞-categories is dualizable when considered as an object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. In particular for every small ∞-category X the ∞-category $\mathcal{P}(X) \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of presheaves is a dualizable object. But for every dualizble object in an ∞-category and every endomorphism one can form a trace. In our case an endomorphism is a colimit preserving functor $F : \mathcal{P}(X) \to \mathcal{P}(X)$ and the trace is then a colimit preserving functor from the tensor unit to itself. The tensor unit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is Sp, the ∞-category of spectra. Thus an endomorphism is essentially the same as a spectrum. Therefore we get that the trace of F is given by a spectrum. One can show that this trace is equivalent to $\mathrm{THH}(F)$ as defined above. Then the equivalence $\mathrm{THH}(F \circ G) \simeq \mathrm{THH}(G \circ F)$ is literally the cyclic invariance of the trace. It is a bit tricky in this approach to get the functoriality of the trace as we need it (however, see [BN13, Section 2.4 and 2.5] for nice discussion of that).

Proposition 6.5. *For every trace theory T and every functor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ there is a C_n-action on $T(F \circ \ldots \circ F) \simeq T(F, \ldots, F)$ which extends to a $\mathbb{T}$-action for $F = \mathrm{id}$.*

Proof. The first claim is obvious since $T(F \circ \ldots \circ F)$ is by the trace property equivalent to $T(F, \ldots, F)$ which has a C_p-action since the object $(F, \ldots, F) \in \Lambda^{\mathrm{st}}$ carries a C_p-action. For the second we observe that $T(\mathrm{id}_{\mathcal{C}})$ is equivalent to the colimit of the cyclic diagram $[n] \mapsto T(\mathrm{id}_{\mathcal{C}}, \ldots, \mathrm{id}_{\mathcal{C}})$ (with n-identities) since all structure maps are equivalences. Thus it gets an induced $\mathbb{T}$-action as the realization of a cyclic object. By subdividing this cyclic object we see that the action is compatible with the one on $T(\mathrm{id}, \ldots, \mathrm{id})$. □

Proposition 6.6. *For every trace theory $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ the induced functor*

$$T : \mathrm{Cat}_\infty^{\mathrm{stab}} \to \Lambda^{\mathrm{st}} \to \mathrm{Sp}$$

is Morita invariant, that is for an exact functor $F : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories which is an equivalence after idempotent completion the induced map $T(\mathcal{C}) \to T(\mathcal{D})$ is an equivalence.

Proof. Being an equivalence after idempotent completion means that we have an inverse functor $G : \mathrm{Ind}\mathcal{D} \to \mathrm{Ind}\mathcal{C}$ such that $G \circ F \simeq \mathrm{id}_{\mathrm{Ind}\mathcal{C}}$ and $F \circ G \simeq \mathrm{id}_{\mathrm{Ind}\mathcal{D}}$. We then have

$$T(\mathcal{C}, \mathrm{id}) \simeq T(\mathcal{C}, G \circ F) \simeq T(\mathcal{D}, F \circ G) \simeq T(\mathcal{D}, \mathrm{id})$$

which shows the claim. □

We recall that a Verdier sequence of stable ∞-categories is a sequence of stable ∞-categories $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ such that the composition is the zero functor (this is a property and not extra structure) and such that it is a fibre and cofibre sequence in $\mathrm{Cat}_\infty^{\mathrm{st}}$.

Proposition 6.7. *Let $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ be a stable trace theory. Then for every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ of stable ∞-categories the induced sequence*

$$T(\mathcal{C}) \to T(\mathcal{D}) \to T(\mathcal{E})$$

is a fibre sequence of spectra.

Proof. The induced sequence

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{i} \mathrm{Ind}(\mathcal{D}) \xrightarrow{p} \mathrm{Ind}(\mathcal{E})$$

is a split Verdier sequence, that is there are right adjoints to R_p to p and R_i to i such that the unit $\mathrm{id} \to R_i \circ i$ as well as the counit $p \circ R_p \to \mathrm{id}$ are equivalences (these exist by the adjoint functor theorem). Then we get a cofibre sequence of functors

$$i \circ R_i \to \mathrm{id}_{\mathrm{Ind}(\mathcal{D})} \to R_p \circ p$$

using the properties of a Verdier sequence. [8] Now for every functor $T : \Lambda^{\mathrm{st}}$ there is a diagram

$$\begin{array}{ccccc}
T(\mathcal{C}) & \longrightarrow & T(\mathcal{D}) & \longrightarrow & T(\mathcal{E}) \\
\downarrow \simeq & & \downarrow = & & \downarrow \simeq \\
T(R_i \circ i) & \longrightarrow & T(\mathrm{id}_{\mathrm{Ind}(\mathcal{D})}) & \longrightarrow & T(p \circ R_p) \\
\downarrow & & \downarrow & & \downarrow \\
T(R_i, i) & \longrightarrow & T(\mathrm{id}_{\mathrm{Ind}(\mathcal{D})}, \mathrm{id}_{\mathrm{Ind}(\mathcal{D})}) & \longrightarrow & T(p, R_p) \\
\uparrow & & \uparrow & & \uparrow \\
T(i \circ R_i) & \longrightarrow & T(\mathrm{id}_{\mathrm{Ind}(\mathcal{D})}) & \longrightarrow & T(R_p \circ p)
\end{array}$$

which commutes since the corresponding diagram in Λ^{st} commutes as can be easily checked. In fact it commutes with the respective nullhomotopies of the horizontal lines, which can be seen by putting in every line an additional object $T(0_{\mathrm{Ind}(\mathcal{D})})$ respectively $T(0_{\mathrm{Ind}(\mathcal{D})}, 0_{\mathrm{Ind}(\mathcal{D})})$ so that every line becomes square (but which we do not want to draw to keep the diagram simpler). Now if T is a trace theory then all the horizontal maps are equivalences and if T is stable then the lower vertical sequence is a cofibre sequence. □

One can get a more general result for localization sequences where coefficients or more generally cyclic graphs are allowed we will discuss that in the next section.

Corollary 6.8. *For every stable ∞-category* $\mathrm{THH}(\mathcal{C}) = \mathrm{THH}(\mathrm{id}_{\mathcal{C}})$ *carries canonically a $\mathbb{T}$-action. For every functor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ the spectrum* $\mathrm{THH}(F^p) \simeq \mathrm{THH}(F, \dots, F)$ *carries a C_p-action. Moreover* THH *is Morita invariant and localizing* [9].

We now can form the spectrum $\mathrm{THH}(\vec{F}, \dots, \vec{F})^{tC_p}$. This assignment again forms a functor $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ in a non-trivial way (refer to the discussion above).

Proposition 6.9. *The functor* $\vec{F} \mapsto \mathrm{THH}(\vec{F}, \dots, \vec{F})^{tC_p}$ *is a stable trace theory.*

Proof. Clearly it is a trace theory since for every contraction in F the induced morphism can (before taking Tate) be written as an p-fold composition of contractions and is thus an equivalence. For stability we use the usual fact about Tate of a multilinear functor being exact. □

[8]Such a situation is called a semi-orthogonal decomposition, one should think of $\mathrm{Ind}(\mathcal{E})$ as local objects for a Bousfield localization and $\mathrm{Ind}(\mathcal{C})$ as the acyclic objects.

[9]Localizing for a functor $\mathrm{Cat}_\infty \to \mathrm{Sp}$ means that it sends Verdier sequence to fibre sequences of spectra. Under the additional assumption of Morita invariance this is equivalent to the notion of localizing discussed in [BGT13, Definition 8.1]

Now we get an induced transformation of functors $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ as in the diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{THH} \\ \downarrow & & \downarrow{\scriptstyle \exists!\, \varphi_p} \\ E^{hC_p} & \longrightarrow & \mathrm{THH}^{tC_p} \end{array} .$$

where the left hand map is given in ... After evaluation on $\mathcal{C}$ this gives us a $\mathbb{T}$-equivariant map

$$\varphi_p : \mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})^{tC_p} .$$

for every stable ∞-category $\mathcal{C}$. This shows that $\mathrm{THH}(\mathcal{C})$ is a cyclotomic spectrum. But the transformation is more general since it also gives some information for THH with coeffiicients: for an endofunctor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ we get a map

$$\mathrm{THH}(F) \to \mathrm{THH}(F, \ldots, F)^{tC_p} \simeq \mathrm{THH}(F \circ \ldots \circ F)^{tC_p}$$

generalizing the cyclotomic Frobenius. This extension of the Frobenius will be used in Section **??** to generalize the Definition of TR to a setting with coefficients.

Corollary 6.10. *The functor* $\mathrm{THH} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ *extends to a functor*

$$\mathrm{THH} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{CycSp}$$

where CycSp *is the* ∞*-category of cyclotomic spectra. This functor is Morita invariant and localizing (i.e. sends Morita equivalences to equivalences and Verdier sequences to cofibre sequences). Therefore also the composite*

$$\mathrm{TC} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{CycSp} \to \mathrm{Sp}$$

is Morita invariant and localizing.

At this point we could directly deduce as in [BGT13, Section 10] that there is a natural transformation from the K-theory functor $K \to \mathrm{TC}$, since by construction we have a natural transformation of the functor $\Sigma^\infty(-)^\sim : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$ to TC and K-theory has a universal property. This is a 'construction' of the cyclotomic trace. We will instead give in the next sections a more highly structured version of the trace and deduce the existence of the classical trace from that.

Remark 6.11. A trace theory is by definition a functor $\Lambda^{\mathrm{st}}[\mathrm{coCart}^{-1}] \to \mathrm{Sp}$ where $\Lambda^{\mathrm{st}}[\mathrm{coCart}^{-1}]$ denotes the Dwyer-Kan localization of Λ^{st} at the coCartesian edges. But this Dwyer-Kan localization is just the colimit over the classifying functor

$$\chi : \Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty \qquad [n]_\Lambda \mapsto \mathrm{Stab}^\flat_{/\mathrm{N}\mathbb{T}_n} .$$

We can should think of $\mathrm{Stab}^\flat_{/\mathrm{N}\mathbb{T}_n}$ as a lax version of the category of functors

$$\mathrm{Fun}(\mathrm{N}\mathbb{T}_n, \mathrm{Cat}_\infty^{\mathrm{st}})$$

A more preise statement is that the latter functor category is equivalent to the subcategory of $\mathrm{Stab}^\flat_{/\mathrm{N}\mathbb{T}_n}$ consisting of the coCartesian fibrations and maps that preserve coCartesian lifts. So we are taking a colimit over a cyclic object in Cat_∞ that looks very much like the unstable cyclic Bar constriction of Section 3.1. The colimit over Λ for a cyclic object is actually the homotopy orbits for the canonical $\mathbb{T}$-action on the geometric realization of the cyclic object. This together tells us that we should think of $\Lambda^{\mathrm{st}}[\mathrm{coCart}^{-1}]$ as a 2-categorical version of THH applied to the $(\infty, 2)$-category $\mathrm{Cat}_\infty^{\mathrm{st}}$.

7. Additivity and localizing theories

In this section we want to generalize the property of having localization sequences that we have established for every trace theory in Proposition 6.7 to arbitrary coefficients. We say that a sequence

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$$

of stable flat fibrations over $\mathrm{N}\mathbb{T}_n$ (considered as a sequence in Λ^{st} over $[n]_\Lambda \in \Lambda^{\mathrm{op}}$) is a Verdier sequence if for every object $k \in \mathrm{N}\mathbb{T}_n$ the sequence

$$\mathcal{X}_k \to \mathcal{Y}_k \to \mathcal{Z}_k$$

is a Verdier sequence of stable ∞-categories and the maps $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ are strict. We say that a Verdier sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ in Λ^{st} is split if for every object $k \in \mathrm{N}\mathbb{T}_n$ the sequence

$$\mathcal{X}_k \to \mathcal{Y}_k \to \mathcal{Z}_k$$

is split Verdier, i.e. the functor $\mathcal{Y}_k \to \mathcal{Z}_k$ (equivalently the morphism $\mathcal{X}_k \to \mathcal{Y}_k$) admits a right adjoint.

Example 7.1. A Verdier sequence $\mathcal{C} \to \mathcal{D}\mathcal{E}$ of stable ∞-categories is also a Verdier sequence when considered as a sequence in Λ^{st}. More precisely that is the seuqence

$$\mathcal{C} \times \mathrm{N}\mathbb{T}_1 \to \mathcal{D} \times \mathrm{N}\mathbb{T}_1 \to \mathcal{E} \times \mathrm{N}\mathbb{T}_1 \ .$$

More generally, if we have a Verdier sequence of stable ∞-categories $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ and a diagram

$$\begin{array}{ccccc} \mathrm{Ind}\mathcal{C} & \xrightarrow{\mathrm{Ind}(i)} & \mathrm{Ind}\mathcal{D} & \xrightarrow{\mathrm{Ind}(p)} & \mathrm{Ind}\mathcal{E} \\ \downarrow F & & \downarrow G & & \downarrow H \\ \mathrm{Ind}\mathcal{C} & \xrightarrow{\mathrm{Ind}(i)} & \mathrm{Ind}\mathcal{D} & \xrightarrow{\mathrm{Ind}(p)} & \mathrm{Ind}\mathcal{E} \end{array}$$

that commutes, i.e. the natural transformations filling the squares are equivalences. Then we get an induced Verdier sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ over $\mathrm{N}\mathbb{T}_1$ in Λ^{st} where $\mathcal{X}$ is classifed by the bimodule F, $\mathcal{Y}$ by G and $\mathcal{Z}$ by H. For $F = \mathrm{id}$, $G = \mathrm{id}$ and $H = \mathrm{id}$ this reduces to the previous example.

In view of the last example (with identities) the following Proposition is a generalization of Proposition 6.7 above.

Proposition 7.2. *For every stable trace theory $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ and every Verdier sequence $X \to Y \to Z$ the induced sequence*

$$T(X) \to T(Y) \to T(Z)$$

is a cofibre sequence of spectra.

Proof. The proof proceeds exactly as the proof of Proposition 6.7. First of all, we note that for a Verdier sequence in Λ^{st} written as

$$(F_1, \dots, F_n) \to (G_1, \dots, G_n) \to (H_1, \dots, H_n)$$

the induced sequence

$$(F_1 \circ \dots \circ F_n) \to (G_1 \circ \dots \circ G_n) \to (H_1 \circ \dots \circ H_n)$$

is a Verdier sequence in Λ^{st} as follows directly from the definition. Since the diagram

$$\begin{array}{ccccc} T(F_1,\ldots,F_n) & \longrightarrow & T(G_1,\ldots,G_n) & \longrightarrow & T(H_1,\ldots,H_n) \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ T(F_1\circ\ldots\circ F_n) & \longrightarrow & T(G_1\circ\ldots\circ G_n) & \longrightarrow & T(H_1\circ\ldots\circ H_n) \end{array}$$

is commutative and the vertical maps are equivalence we can this reduce to the claim to Verdier sequences in Λ^{st} over $\mathrm{N}\mathbb{T}_1$. These are of the form described in Example 7.1 above. Thus let us assume we have such a Verdier squence

$$\begin{array}{ccccc} \mathrm{Ind}\mathcal{C} & \xrightarrow{i} & \mathrm{Ind}\mathcal{D} & \xrightarrow{p} & \mathrm{Ind}\mathcal{E} \\ \downarrow{F} & & \downarrow{G} & & \downarrow{H} \\ \mathrm{Ind}\mathcal{C} & \xrightarrow{i} & \mathrm{Ind}\mathcal{D} & \xrightarrow{p} & \mathrm{Ind}\mathcal{E} \end{array} \tag{4}$$

(where we write i and p instead of $\mathrm{Ind}(i)$ and $\mathrm{Ind}(p)$ to simplify the notation. Just from the fact that $\mathcal{C} \to \mathcal{D}\mathcal{E}$ is a Verdier sequence of stable ∞-categories we see as in the proof of Proposition 6.7 that we have a fibre sequence of functors

$$i \circ R_i \to \mathrm{id}_{\mathrm{Ind}(\mathcal{D})} \to R_p \circ p$$

where R_i and R_p are the right adjoints to i and p. Applying the functor G to this sequence we get an induced sequence

$$G \circ i \circ R_i \to G \to G \circ R_p \circ p$$

which induces then a fibre sequence

$$T(G \circ i \circ R_i) \to T(G) \to T(G \circ R_p \circ p)\ . \tag{5}$$

We now use the commutativity of the diagram (4) and the trace property for T to compute the first term:

$$T(G \circ i \circ R_i) \simeq T(i \circ F \circ R_i) \simeq T(R_i \circ i \circ F) \simeq T(F)\ .$$

Similarly we get

$$T(G \circ R_p \circ p) \simeq T(R_p \circ p \circ G) \simeq T(R_p \circ H \circ p) \simeq T(p \circ R_p \circ H) \simeq T(H)\ .$$

Moreover a straightforward variant of the diagram given in the proof of Proposition 6.7 shows that under these equivalences the sequence (5) is equivalent to the sequence $T(F) \to T(G) \to T(H)$ in question. □

Definition 7.3. *A theory* $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is called* additive *if it sends sends all the objects* $0 \times \mathrm{N}\mathbb{T}_n \to \mathrm{N}\mathbb{T}_n \in \Lambda^{\mathrm{st}}$ *to a zero object in spectra and split Verdier sequnces to fibre sequences of spectra. it is called* localizing *if it moreover sends arbitrary Verdier sequences to fibre sequences in* Sp.

Remark 7.4. There is in principle also a way of defining localizing theories without the assumption that $0 \times \mathrm{N}\mathbb{T}_n \to \mathrm{N}\mathbb{T}_n \in \Lambda^{\mathrm{st}}$ is sent to 0. Then one has to impose that for a Verdier sequence considered as a square where one corner is zero that the induced square is Cartesian. We will not need this generality here.

Note that if a theory T sends $0 \times \mathrm{N}\mathbb{T}_n \to \mathrm{N}\mathbb{T}_n \in \Lambda^{\mathrm{st}}$ to zero, this does not imply that T is reduced in the sense of Definition 5.4. A counterexample is given by K-theory of endomorphisms, see the remarks at the begining of Section 9. But the converse is true: if T is reduced then $T(0 \times \mathrm{N}\mathbb{T}_n \to \mathrm{N}\mathbb{T}_n) \simeq 0$.

Finally we shall a variant of the $S_\bullet$ construction from K-theory adapted to our category Λ^{st}. This will be relevant in establishing the universal property if cyclic K-theory in the next section. Since it is of rather technical nature we advise the reader to skip it on first reading.

Construction 7.5. Let $\mathcal{X} \to \mathrm{N}\mathbb{T}_n \in \Lambda^{\mathrm{st}}$ be a flat stable fibration. We define a new flat stable fibration $S_k\mathcal{X} \to \mathrm{N}\mathbb{T}_n$ as follows: the ∞-category $S_k\mathcal{X}$ is a full subcategory of the ∞-category P obtained as the pullback of simplicial sets

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Fun}(\mathrm{Arr}(\Delta^k), \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{N}\mathbb{T}_n & \xrightarrow{\text{const}} & \mathrm{Fun}(\mathrm{Arr}(\Delta^n), \mathrm{N}\mathbb{T}_n) \end{array}$$

Then $S_k\mathcal{X} \subseteq P$ is the full subcategory consisting of those functors $F : \mathrm{Arr}(\Delta^k) \to \mathcal{X}_i$ (where $i \in \mathrm{N}\mathbb{T}_n$) that lie in the usual S_k-construction of $\mathcal{X}_i$, i.e. that have the following properties: $F(\mathrm{id}_y) \simeq 0$ for each $y \in \Delta^k$ and the square

$$\begin{array}{ccc} F(x < y) & \longrightarrow & F(x < z) \\ \downarrow & & \downarrow \\ F(y = y) & \longrightarrow & F(y < z) \end{array}$$

is Cartesian for each triple $x < y < z$ in Δ^k.

Informally $S_k\mathcal{X}$ is given as follows: we write $\mathcal{X}$ as $(F_1, \ldots, F_n)$, then $S_k\mathcal{X}$ is $(F_1^{S_k}, \ldots, F_n^{S_k})$ where

TODO!

Lemma 7.6. *For every additive theory $T : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ the induced functor ...*

8. K-theory of endomorphisms

In this section we want to give the definition of cyclic K-Theory. Let us first review K-theory of endomorphisms. For a ring R we can form an ordinary category $\mathcal{E}$ whose objects are pairs consting of a finitely generated projective R-module M togerther with an endomorphism $f : M \to M$. This category has a notion of short exact sequences (but note that not every of these sequences is split, since the split that exists on underlying modules might not be compatible with the endorphisms). Then we can take exact K-theory of this category and obtain a spectrum

$$\mathrm{K}^{\mathrm{End}}(R) := \mathrm{K}(\mathcal{E}) \ .$$

We will generalize and study this in the setting of stable ∞-categories below but let us first give the reader a bit of a feeling for $\mathrm{K}^{\mathrm{End}}(R)$. More precisely we will focus on $\mathrm{K}_0^{\mathrm{End}}(R) := \pi_0 \mathrm{K}^{\mathrm{End}}(R)$. By construction $\mathrm{K}_0^{\mathrm{End}}(R)$ is the abelian group generated by isomorphism classes of endomorphism $f : M \to M$ subject to the relations $f = f' + f''$ whenever there is a diagram

$$\begin{array}{ccccc} M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' \end{array}$$

of projective R-modules such that $M' \to M \to M''$ is exact.

Example 8.1. Assume that $R = k$ is an algebraically closed field. Then every endomorphism f has a Jordan normal form which is an upper triangular matrix. Thus up to filtration f agrees with its diagonal part, therefore the element $[f] \in \mathrm{K}_0^{\mathrm{End}}(k)$ is represented by a diagonal matrix. The only invariant of f that is left are therefore the diagonal entries $\lambda_1, \ldots, \lambda_n$, but not their ordering, i.e. the set of Eigenvalues and the dimension of the respective Jordan blocks . We can consider the associated element $\sum_{i=1}^n \lambda_i$ in the free abelian group $\mathbb{Z}[k]$. Really $\mathbb{Z}[k]$ is the monoid ring with respect to the multiplicative monoid of k if one takes multiplicative structures on K-theory of endomorphisms into account. This defines an isomorphism

$$\mathrm{K}_0^{\mathrm{End}}(k) \simeq \mathbb{Z}[k] \ .$$

A slightly different way of phrasing the result is to consider the characteristic polynomial

$$\chi_f(t) := \det\left(\mathrm{id} - tf\right) = \prod_{i=1}^{n}(1 - t\lambda_i) \in k[t] \ .$$

(note that we define this with t at a different spot than usual). This assignment sends short exact sequence of endomorphism to products so that it induces a map

$$\chi : \mathrm{K}_0^{\mathrm{End}}(k) \to k(t)$$

where $k(t)$ is the field of rational functors. In fact χ surjects onto the multiplicative groups $W_{\mathrm{rat}}(k)$ of rational functions which are quotients of polynomials with constant term 1. This group can be considered as a subgroup of the multiplicative group $t + tk[[t]]$ of power series with constant term 1 which is the underlying additive group of the ring $W(k)$ of Witt vectors. The map χ sends tensor product of endomorphisms to the product in the Witt vectors and one can show that $W_{\mathrm{rat}}(k)$ is a subring. The map χ is of course not injective since it does not see the eigenvalues 0. But the combination

$$(\chi, \dim) : \mathrm{K}_0^{\mathrm{End}}(k) \to W_{\mathrm{rat}}(k) \times \mathbb{Z}$$

is an isomorphism. Note that this is with the above description compatible by noting that $W_{\mathrm{rat}}(k) \simeq \mathbb{Z}[k^\times]$ and $\mathbb{Z}[k] \simeq \mathbb{Z}[k^\times] \times \mathbb{Z}$ where the latter isomorphism uses the projection in the first factor and the augmentation in the second factor.

Example 8.2. For a general commutative ring R we still have the map

$$(\chi, \dim) : \mathrm{K}_0^{\mathrm{End}}(R) \to W_{\mathrm{rat}}(R) \oplus \mathrm{K}_0(R)$$

and it is a theorem of Almkvist that this is always an isomorphism [Alm74].

Lets now come back to formal properties of K-theory of endomorphisms. Note that instead of working with endomorphisms of projective R-modules one could also pass to the derived ∞-category $\mathcal{D}^{\mathrm{perf}}(R)$ of perfect complexes and form the category $\mathrm{End}(\mathcal{D}^{\mathrm{perf}}(R))$ in the sense of Section 4. This is the ∞-category whose objects are just perfect chain complexes over R equipped with an endomorphism. Equivalently it is the functor category $\mathrm{Fun}(\mathrm{N}\mathbb{T}_1, \mathcal{D}^{\mathrm{perf}}(R))$ which is a stable ∞-category and we can take K-theory of this stable ∞-category[10] . It is a theorem of Blumberg-Gepner-Tabuada [BGT16] that these two possible definitions of $\mathrm{K}^{\mathrm{End}}$ agree, i.e. that

$$\mathrm{K}^{\mathrm{End}}(R) \simeq \mathrm{K}(\mathrm{End}(\mathcal{D}^{\mathrm{perf}}(R))) \ .$$

[10]Here K-theory always refers to connective K-theory. There is of course also a non-connective version but that will only play a minor role here.

This generalized the classical Gillet-Waldhausen result that K-theory of projective R-modules is equivalent to K-theory of the stable ∞-category $\mathcal{D}^{\mathrm{perf}}(R)$ of perfect complexes (which was initial proven in a slightly different language of course).

Definition 8.3. *We define a functor* $\mathrm{K}^{\mathrm{End}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *as the composition*

$$\Lambda^{\mathrm{st}} \xrightarrow{\mathrm{End}} \mathrm{Cat}_{\infty}^{\mathrm{stab}} \xrightarrow{\mathrm{K}} \mathrm{Sp}$$

where K refers to the connective K-theory spectrum. Since the functor $\mathrm{K} : \mathrm{Cat}_{\infty}^{\mathrm{stab}} \to \mathrm{Sp}$ *comes with a natural transformation* $(-)^{\sim} \to \Omega^{\infty} K$ *the composition of* End *with the adjoint gives us a natural transformation* $E \to \mathrm{K}^{\mathrm{End}}$.

Theorem 8.4. *The theory* $\mathrm{K}^{\mathrm{End}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is additive. Moreover the morphism* $E \to \mathrm{K}^{\mathrm{End}}$ *exhibits it as the universal additive theory under E.*

Proof. TODO . □

9. Cyclic K-theory

In the last section we have seen that $K^{\mathrm{End}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ is additive. We would like it to be a trace theory. Unfortunately it is not a trace theory as we will see soon.

Lemma 9.1. *Assume that we have a trace theory T such that* $T(0 \times \mathrm{N}\mathbb{T}_1 \to \mathrm{N}\mathbb{T}_1) \simeq 0$. *Then T is reduced in the sense of Definition 5.4.*

Proof. For any stable ∞-category $\mathcal{C}$ we consider the following zero functors

$$0_1 : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C} \qquad 0_2 : \mathrm{Ind}\mathcal{C} \to 0 \qquad 0_3 : 0 \to \mathrm{Ind}\mathcal{C}$$

where 0 denotes the zero category, which is equivalent to Ind(0). We have in particular $0_1 \simeq 0_3 \circ 0_2$. We now find for any list $(F_1, \ldots, F_n)$ that contains a zero functor

$$T(F_1, \ldots, F_n) \simeq T(0_1) \simeq T(0_3, 0_2) \simeq T(0_2 \circ 0_3) \simeq T(0 \times \mathrm{N}\mathbb{T}_1 \to \mathrm{N}\mathbb{T}_1) \simeq 0$$

which shows the claim. □

Clearly we have that $K^{\mathrm{End}}(0 \times \mathrm{N}\mathbb{T}_1 \to \mathrm{N}\mathbb{T}_1) \simeq 0$. Thus if K^{End} were a trace theory it had to be reduced. But it is not reduced :we get for any stable ∞-category $\mathcal{C}$ that $\mathrm{K}^{\mathrm{End}}(0_{\mathrm{Ind}\mathcal{C}}) \simeq \mathrm{K}(\mathcal{C})$ since $\mathrm{End}(0_{\mathrm{Ind}\mathcal{C}}) \simeq \mathcal{C}$. We account for this failure by making K^{End} reduced and then it will turn out that the result is in fact a trace theory.

Definition 9.2. *The functor* $\mathrm{K}^{\mathrm{cyc}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is the universal reduced functor under* $\mathrm{K}^{\mathrm{End}}$*, i.e.* $\mathrm{K}^{\mathrm{cyc}} := (\mathrm{K}^{\mathrm{End}})^{\mathrm{red}}$ *with the Notation of Proposition 5.5.*

Note that Proposition 5.5 gives us a concrete formula how to obtain $\mathrm{K}^{\mathrm{cyc}}(F_1, \ldots, F_n)$. In particular for a single functor $F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{C}$ (i.e. the identity) we get that

$$\mathrm{K}^{\mathrm{cyc}}(F) \simeq \mathrm{K}^{\mathrm{End}}(F) / \mathrm{K}(\mathcal{C}) \ .$$

Note that he map for which we take the cofibre $\mathrm{K}(\mathcal{C}) = \mathrm{K}^{\mathrm{End}}(0) \to \mathrm{K}^{\mathrm{End}}(F)$ is given by sending an object to the object with the zero endomorphism.

Example 9.3. For a commutative ring R we have an isomorphism

$$\mathrm{K}_0^{\mathrm{cyc}}(R) \simeq W_{\mathrm{rat}}(R)$$

which immediately follow from the description in Example 8.2.

As in this example, the map by which we take the cofibre is always split by the functor that forgets the endomorphism. Therefore the quotient defining $\mathrm{K}^{\mathrm{cyc}}(F)$ is really splitting off a summand. A similar remark is true for the reduction of an arbitrary functor and an arbitrary list of functors $(F_1, \ldots, F_n)$. The value $\mathrm{K}^{\mathrm{cyc}}(F_1, \ldots, F_n)$ is according to Proposition 5.5 given by the total cofibre of a cube. But it turns out that in this example this can be drastically simplified:

Lemma 9.4. *The value* $\mathrm{K}^{\mathrm{cyc}}(F_1, \ldots, F_n)$ *is equivalent to the cofibre of the map*

$$\mathrm{K}\,\mathcal{C}_1 \times \ldots \times \mathrm{K}\,\mathcal{C}_n \simeq \mathrm{K}^{\mathrm{End}}(0_{\mathrm{Ind}\mathcal{C}_1}, \ldots, 0_{\mathrm{Ind}\mathcal{C}_n}) \to \mathrm{K}^{\mathrm{End}}(F_1, \ldots, F_n)\ .$$

Proof. As mentioned above, there is an a priori formula for $\mathrm{K}^{\mathrm{cyc}}(F_1, \ldots, F_n)$ by the total cofibre of an n-cube. The initial vertex of this cube of spectra is

$$\mathrm{K}^{\mathrm{End}}(0_{\mathrm{Ind}\mathcal{C}_1}, \ldots, 0_{\mathrm{Ind}\mathcal{C}_n})$$

and the terminal vertex is $\mathrm{K}^{\mathrm{End}}(F_1, \ldots, F_n)$. Thus to deduce the result it suffices to show that every edge of this cube which does not end in the terminal vertex is an equivalence. This comes down to checking that all the maps

$$\mathrm{K}^{\mathrm{End}}(0_{\mathrm{Ind}\mathcal{C}_1}, \ldots, 0_{\mathrm{Ind}\mathcal{C}_n}) \to \mathrm{K}^{\mathrm{End}}(F_1, \ldots, F_n)$$

are equivalences as soon as one of the $F_i's$ is zero. We assume without loss of generality that $F_n = 0$. On the level of categories the map in question corresponds to the fully faithful inclusion

$$\mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathrm{End}(F_1, \ldots, F_n)$$

and for $F_n = 0$ the target category is equivalent to ... TODO. □

Now we come to the main result of this section. A variant of this theorem is due to Kaledin in unpublished work (at least the first part, but we are not entirely sure what his precise results are).

Theorem 9.5. *The theory* $\mathrm{K}^{\mathrm{cyc}} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *is an reduced and addtive trace theory. Moreover the morphism* $E \to \mathrm{K}^{\mathrm{End}} \to \mathrm{K}^{\mathrm{cyc}}$ *exhibits it as either of the following:*

(1) *the universal reduced and additive theory under* E
(2) *the universal additive trace theory under* E.

Proof. By construction $\mathrm{K}^{\mathrm{cyc}}$ is reduced. The additivity follows from the fact that $\mathrm{K}^{\mathrm{End}}$ is additive (see Theorem 8.4) as follows: for a given split Verdier sequence

$$(F_1, \ldots, F_n) \to (G_1, \ldots, G_n) \to (H_1, \ldots, H_n) \tag{6}$$

we consider the sequence

$$(0, \ldots, 0) \to (0, \ldots, 0) \to (0, \ldots, 0)$$

where source and target of the zero funcrtors are exactly the same as for the sequence above, In other words: we just replace any of the funtors by 0. Then this is also a split Verdier sequemce and thus also induces a short fibre sequence after applying K^{End}. In fact the latter fact can be seen much easier, since the category of endomorphisms is just a product of the respective fibre sequnces, so that we get a producft of fibre sequence which is a fibre sequence. But according to Lemma 9.4 above we then get $\mathrm{K}^{\mathrm{cyc}}$ applied to the Verdier sequence 6 as the cofibre of these two fibre sequence, this it is a fibre sequence as well.

To show that $\mathrm{K}^{\mathrm{cyc}}$ is a trace theory we have to prove that for two functors $F : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ and $G : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ we have that the ma[

$$\mathrm{K}^{\mathrm{cyc}}(\ldots, F, G, \ldots) \xrightarrow{\simeq} \mathrm{K}^{\mathrm{cyc}}(\ldots, F \circ G, \ldots)$$

is an equivalence where the dots indicate some other functors. For simplicity of notation we omit these other functors, assume that $\mathcal{E} = \mathcal{C}$ and show that

$$\mathrm{K}^{\mathrm{cyc}}(F, G) \simeq \mathrm{K}^{\mathrm{cyc}}(F \circ G)$$

but the general proof works exactly the same. Recall that by definition and Lemma 9.4 we have a diagram

$$\begin{array}{ccc}
K(\mathcal{D}) \oplus K(\mathcal{C}) & \longrightarrow & \mathrm{K}(\mathcal{D}) \\
\downarrow & & \downarrow \\
K^{\mathrm{End}}(F, G) & \longrightarrow & K^{\mathrm{End}}(FG) \\
\downarrow & & \downarrow \\
\mathrm{K}^{\mathrm{cyc}}(F, G) & \longrightarrow & \mathrm{K}^{\mathrm{cyc}}(F \circ G)
\end{array}$$

in which the vertical lines are fibre sequences. We want to show that the lower horizontal map is an equivalence. This is equivalent to showing the the upper square is a pullback which is equivalent to showing that the induced map from $\mathrm{K}(\mathcal{C})$ to the fibre of $\mathrm{K}^{\mathrm{End}}(F, G) \to \mathrm{K}^{\mathrm{End}}(FG)$ is an equivalence. But this follows from the fact that this map is induced from the Verdier sequence

$$\mathcal{C} \to \mathrm{End}(F, G) \to \mathrm{End}(FG)$$

whose first map takes $c \in \mathcal{C}$ to the pair $(c, 0)$ with the zero morphisms and whose second map forgets the object in $\mathcal{C}$ and composes the morphisms.

Finally we want to show the universal properties of $\mathrm{K}^{\mathrm{cyc}}$: the universal property 1 is true by the fact that $\mathrm{K}^{\mathrm{End}}$ (Theorem 8.4) is the universal additive theory and that $\mathrm{K}^{\mathrm{cyc}}$ is defined to be the associated reduced theory (which is still additive as shown above).

The universal property (2) then follows from the first and Lemma 9.1 since every additive trace theory is automatically reduced.

□

Corollary 9.6. *For every stable ∞-category $\mathcal{C}$ there is a canonical action of $\mathbb{T}$ on $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C})$. Moreover since $(\mathrm{K}^{\mathrm{cyc}})^{hC_p}$ is an additive theory as we get for every integer p (not necessarily a prime here) a canonical map ψ_p in the diagram*

$$\begin{array}{ccc}
E & \longrightarrow & \mathrm{K}^{\mathrm{cyc}} \\
\downarrow & & \Big\downarrow {\scriptstyle \exists! \, \psi_p} \\
E^{hC_p} & \longrightarrow & (\mathrm{K}^{\mathrm{cyc}})^{hC_p}
\end{array} .$$

These maps give $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C})$ the structure of a cyclotomic spectrum with Frobenius lifts.

The author learned the existence of the $\mathbb{T}$-action on $\mathrm{K}^{\mathrm{End}}(\mathcal{C})$ from a talk of Dimity Kaledin [**?**] and also wants to thank Peter Scholze for pointing this out to him. It is very surprising that this $\mathbb{T}$-action exists.

Remark 9.7. There is a variant of the category $\mathrm{End}(\mathcal{C})$ for an arbitrary stable ∞-category called the category of automorphisms (or better autoequivalences) which is defined as

$$\mathrm{Aut}(\mathcal{C}) := \mathrm{Fun}(\mathbb{T}, \mathcal{C})$$

where $\mathbb{T} \simeq B\mathbb{Z}$ here denotes the simplicial circle considered as a Kan complex (not its classifying space). Thus an object in $\mathrm{Aut}(\mathcal{C})$ is given by an object of $\mathcal{C}$ together with an automorphism. Clearly $\mathrm{Aut}(\mathcal{C})$ is a stable ∞-category and a full subcategory of $\mathrm{End}(\mathcal{C})$. The functor $\mathrm{Aut}(-)$ can also be extended to Λ^{st} by taking the full subcategory of $\mathrm{End}(F_1, \ldots, F_n)$ consisting of sequences of objects $c_1, \ldots, c_n$ together with equivalences $c_i \xrightarrow{\simeq} F_i c_{i+1}$. Then $\mathrm{K}^{\mathrm{Aut}}(\mathcal{C})$ is defined as K-theory of the stable ∞-category $\mathrm{Aut}(\mathcal{C})$ and more generally we get a functor as the composite

$$\mathrm{K}^{\mathrm{Aut}} : \Lambda^{\mathrm{st}} \xrightarrow{\mathrm{Aut}} \mathrm{Cat}^{\mathrm{st}}_{\infty} \xrightarrow{\mathrm{K}} \mathrm{Sp} \ .$$

Now for a stable ∞-category $\mathcal{C}$ there is an action of $\mathbb{T}$ directly on the ∞-category $\mathrm{Aut}(\mathcal{C}) = \mathrm{Fun}(\mathbb{T}, \mathcal{C})$ given by multiplication in the source. Informally a $\mathbb{T}$-action on an ∞-category is given by the choice of an automorphism for every object of the ∞-category [11]. So roughly the action on $\mathrm{Aut}(\mathcal{C})$ is given by equipping every object, which is an object of c with an autormorphism f with the automorphisms f itself. Thus $\mathrm{Aut}(\mathcal{C})$ is exactly taylored to allow for such a $\mathbb{T}$-action . Then by functoriality of the K-theory functor we get an induced $\mathbb{T}$-action on $\mathrm{K}^{\mathrm{Aut}}(\mathcal{C})$. The $\mathbb{T}$-action on $\mathrm{Aut}(\mathcal{C})$ clearly does not extend to a $\mathbb{T}$-action on $\mathrm{End}(\mathcal{C})$ so a similary trick can not be used to directly get a $\mathbb{T}$-action on $\mathrm{K}^{\mathrm{End}}(\mathcal{C})$[12]. This fact makes the existence of the $\mathbb{T}$-action in $\mathrm{K}^{\mathrm{cyc}}$ even more mysterious to the author.

There is another interesting aspect of K-theory of automorphisms: by construction there are natural transformations $\mathrm{K}^{\mathrm{Aut}} \to \mathrm{K}^{\mathrm{End}} \to \mathrm{K}^{\mathrm{cyc}}$ and one can show that for a field k the induced map

$$\mathrm{K}^{\mathrm{Aut}}(k) \to \mathrm{K}^{\mathrm{cyc}}(k)$$

is an equivalence of spectra where $\mathrm{K}^{\mathrm{Aut}}(k) = \mathrm{K}^{\mathrm{Aut}}(\mathcal{D}^{\mathrm{perf}}(k))$ as usual. In general this is not true and we believe that $\mathrm{K}^{\mathrm{Aut}}$ is not even a trace theory.

We not want to establish a final piece of structure on the spectrum $K^{\mathrm{cyc}}(\mathcal{C})$ for a stable ∞-category $\mathcal{C}$ and this will definitely not have an anlog for K^{cyc} applied to a general object $(F_1, \ldots, F_n) \in \Lambda^{\mathrm{st}}$. Recall that by the last result $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C})$ is a cyclotomic spectrum. This means that we can take its topological cyclic homology. But in fact, $\mathrm{K}^{\mathrm{cyc}}$ carries more stucture, namely the Frobenius lifts which gives us an action of the monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$. In such a case, i.e. for a general spectrum with a $\mathbb{T} \rtimes \mathbb{N}_{>0}$ one has a refinement of TC, namely the homotopy fixed points for this action and there is a canonical map

$$X^{h(\mathbb{T} \rtimes \mathbb{N}_{>0})} \to \mathrm{TC}(X)$$

where TC is taken with respect to the induced cyclotomic structure.

[11]But note that this has to satisfy some serious coherences, in particular these morphisms have to commute with itself in an $\mathbb{E}_2$-way which is condition the author has problems imagining or formulating an informal way.

[12]One could try to argue that there is an action by category $B\mathbb{N}$ instead which is a 2-categorical concept. But since the K-theory functor is not a 2-functor (it takes the classifying space first which definitely discards all non-invertible morphism) this can to the best knowledge of the author not be used to get a $\mathbb{T}$-action on the spectrum $\mathrm{K}^{\mathrm{End}}(\mathcal{C})$.

Proposition 9.8. *For every stable ∞-category $\mathcal{C}$ there is a canonical map*

$$K(\mathcal{C}) \to K^{\mathrm{cyc}}(\mathcal{C})^{h(\mathbb{T}\rtimes\mathbb{N}_{>0})}$$

induced from the functor which takes an object $M \in \mathcal{C}$ to the object $(M, \mathrm{id}_M) \in \mathrm{End}(\mathcal{C})$. Moreover this map is natural in $\mathcal{C}$ i.e. extends to a natural transformation of functors $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$.

Proof. TODO □

10. The cyclotomic trace

In this section we want to construct a variant of the cyclotomic trace which is more highly structured than the usual variant. As a first step we note that until now we have defined two important functors

$$K^{\mathrm{cyc}}, \mathrm{THH} : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$$

both of which admit transformations $E \to K^{\mathrm{cyc}}$ and $E \to \mathrm{THH}$ which satisfy universal properties: K^{cyc} is universal among additive theories under E and THH is universal among stable trace theories under E. Since every stable trace theory is additive as shown in 7.2 we immediately get the following result:

Theorem 10.1. *There is a unique natural transformation*

$$\mathrm{tr} : K^{\mathrm{cyc}} \to \mathrm{THH}$$

compatible with the maps from E. Moreover for every prime p the induced diagrams

$$\begin{array}{ccc} K^{\mathrm{cyc}} & \xrightarrow{\quad\quad\mathrm{tr}\quad\quad} & \mathrm{THH} \\ \downarrow{\psi_p} & & \downarrow{\varphi_p} \\ (K^{\mathrm{cyc}})^{hC_p} \xrightarrow{\mathrm{can}} (K^{\mathrm{cyc}})^{tC_p} & \xrightarrow{\mathrm{tr}^{tC_p}} & \mathrm{THH}^{tC_p} \end{array}$$

canonically commute.

Remark 10.2. While the way we have proven the Theorem is rather abstract, it does give us a somewhat concrete way of describing the trace: TODO

Recall that the maps ψ_p make $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C})$ to a cyclotomic spectrum with Frobenius lifts for every stable ∞-category $\mathcal{C}$. In particular they extend to an action of the monoid $\mathbb{T} \rtimes \mathbb{N}_{>0}$. The underlying cyclotomic spectrum has the p-th Frobenius given by the composition

$$\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}) \xrightarrow{\psi_p} \mathrm{K}^{\mathrm{cyc}}(\mathcal{C})^{hC_p} \xrightarrow{\mathrm{can}} \mathrm{K}^{\mathrm{cyc}}(\mathcal{C})^{tC_p} \ .$$

When we refer to $K^{\mathrm{cyc}}(\mathcal{C})$ as a cyclotomic spectrum we always mean this induced cyclotomic structure.

Corollary 10.3. *For every stable ∞-category $\mathcal{C}$ we get an induced map*

$$\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})$$

of cyclotomic spectra and thus using the map of Proposition 9.8 as a composite

$$\mathrm{K}(\mathcal{C}) \to (\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}))^{h(\mathbb{T}\rtimes\mathbb{N}_{>0})} \to \mathrm{TC}(\mathrm{K}^{\mathrm{cyc}}(\mathcal{C})) \to \mathrm{TC}(\mathrm{THH}(\mathcal{C}))$$

the trace in its ordinary incarnation.

Proof. Clear but we should given argument that its unique. □

We can now also use he universal properties that we have established to deduce the following nice result, initially proven by Dundas-McCarty

Corollary 10.4. *The natural transformation*

$$\mathrm{tr} : \mathrm{K}^{\mathrm{cyc}} \to \mathrm{THH}$$

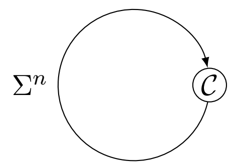
of functors $\Lambda^{\mathrm{st}} \to \mathrm{Sp}$ *exhibits* THH *as the stablilization of* K^{cyc}*, i.e. induces an equivalence* $(K^{\mathrm{cyc}})^{\mathrm{st}} \to \mathrm{THH}$ *where* $(-)^{\mathrm{st}}$ *is the construction described in Proposition 5.5.*

Proof. This follows by noting that $(K^{\mathrm{cyc}})^{\mathrm{st}}$ is a stable trace theory (since stabilization clearly does by the given formula clearly not destroy the property of being a trace theory). Then its initial among those, since its in principle initial among stable trace theories that are additive, but additivity follows from being a stable trace theory as shown in [**?**]. But this universal property is also satisfied by THH. □

Note that we do not only get a very abstract equivalence from the last corollary but a rather concrete formula for THH in terms of K-theory, namely

$$\mathrm{THH}(\mathcal{C}) \simeq \operatorname*{colim}_{n\to\infty} \Omega^n \, \mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, \Sigma^n)$$

where the latter is K^{cyc} evaluated on the object



in Λ^{st} as usual. A similar formula can be obtained for an arbitrary object in Λ^{st}. Also note that it automatically follows that the equivalence

$$(K^{\mathrm{cyc}})^{\mathrm{st}}(\mathcal{C}) \simeq \mathrm{THH}(\mathcal{C})$$

is an equivalence of cyclotomic spectra. We want to end this section by a discussion of the relation of this last result to the a slightly different stabilization of K-theory that was introduced by Waldhausen. This relation is also nicely discussed by Dundas-McCarty [**?**] but we include it for the sake of completeness. It also plays a role in the proof of the Dundas McCarthy theorem relating K-theory and TC as nicely explained in [**?**].

11. Generalized TR theory and a lift of the trace

Define TR and deduce the lift of the trace. This is given by the inclusion of the rational Witt vectors into the Witt vectors.

$$\begin{array}{ccc} \mathrm{TR}^n(\mathcal{C}, F) & \longrightarrow & \mathrm{TR}^{n-1}(\mathcal{C}, F) \\ & & \big\downarrow \\ & & \mathrm{THH}(\mathcal{C}, F, \dots, F)^{tC_p} \end{array}$$

12. The Lindenstrauss-McCarthy theorem

Show that the Goodwillie derivatives (in the Bimodule) of $\mathrm{K}^{\mathrm{cyc}}$ are given by $\mathrm{TR}^n : \Lambda^{\mathrm{st}} \to \mathrm{Sp}$

13. Symmetric monoidal structures

We show that Λ^{st} extends to an ∞-operad over $(\Lambda^{\mathrm{op}})^{\sqcup}$ such that the functor $p : (\Lambda^{\mathrm{st}})^{\otimes} \to (\Lambda^{\mathrm{op}})^{\sqcup}$ is a coCartesian fibration of ∞-operads. In particular all the fibres of $\Lambda^{\mathrm{st}} \to \Lambda^{\mathrm{op}}$ inherit symmetric monoidal strucutures. Moreover THH extends to a lax monoidal functor

$$\mathrm{THH} : (\Lambda^{\mathrm{st}})^{\otimes} \to \mathrm{Sp}^{\otimes}$$

that sends p-coCartesian lifts to coCartesian lifts.

Appendix A. Adjoints and coCartesian fibrations

In this appendix we shall formulate a criterion for certain adjoints. We assume that we have a coCartesian fibration $X \to S$ of ∞-categories and we consider a functor ∞-category $\mathrm{Fun}(X, \mathcal{E})$ for some ∞-category $\mathcal{E}$. Assume that for each $s \in S$ we have a Bousfield localization $L_s : \mathrm{Fun}(X_s, \mathcal{E}) \to \mathrm{Fun}(X_s, \mathcal{E})$. We seek to find a criterion under which we can 'glue' the L_s together to a Bousfield localization $L : \mathrm{Fun}(X, \mathcal{E}) \to \mathrm{Fun}(X, \mathcal{E})$.

Theorem A.1. *Assume that for each edge $s \to s'$ in S the induced map*

$$\mathrm{Fun}(X_{s'}, \mathcal{E}) \to \mathrm{Fun}(X_s, \mathcal{E})$$

sends $L_{s'}$-local objects to L_s-local objects. Then there is a localization

$$L \colon \mathrm{Fun}(X, \mathcal{E}) \to \mathrm{Fun}(X, \mathcal{E})$$

such that for every $s \in S$ the diagram

$$\begin{array}{ccc} \mathrm{Fun}(X, \mathcal{E}) & \xrightarrow{L} & \mathrm{Fun}(X, \mathcal{E}) \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Fun}(X_s, \mathcal{E}) & \xrightarrow{L_s} & \mathrm{Fun}(X_s . \mathcal{E}) \end{array}$$

commutes and such that the L-local objects are precisely those functors $F : X \to \mathcal{E}$ such that each restriction $F\,|_{X_s} : X_s \to \mathcal{E}$ is L_s-local.

Appendix B. An idempotent criterion

Let $\mathcal{C}$ be an ∞-category and assume that we have a functor $L : \mathcal{C} \to \mathcal{C}$ with a natural transformation $\eta : \mathrm{id} \to L$. Is is shown in [Lur09, Proposition 5.2.7.4] that L is a Bousfield localization (i.e. the left adjoint onto a full subcategory $L\mathcal{C} \subseteq \mathcal{C}$) if and only if the two induced natural transformations

$$L(\eta), \eta(L) \colon L \to L^2$$

are equivalences. We shall prove a local criterion of this sort here. This criterion is more or less standard, but we haven't found it spelled out explicitly somewhere and therefore we include a quick proof.

Proposition B.1. *Assume that we have a functor $L : \mathcal{C} \to \mathcal{C}$ with a natural transformation $\eta : \mathrm{id} \to L$ and a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ such that for every object $d \in \mathcal{C}^0$ the map $d \to Ld$ is an equivalence. Assume moreover that for any object $c \in \mathcal{C}$ we have that the two maps $Lc \to LLc$ are equivalent as objects in $\mathcal{C}^{\Delta^1}$.*

If for a given $c \in \mathcal{C}$ the object Lc is in $\mathcal{C}^0$ then the map $c \to Lc$ is initial among all maps under c with target in $\mathcal{C}^0$.

Proof. Consider the full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ consisting of objects $c \in \mathcal{C}$ such that $Lc \in \mathcal{C}^0$. Then L restricts to a functor $\mathcal{C}' \to \mathcal{C}'$. Moreover the map $\eta_{Lc} : Lc \to L^2c$ is an equivalence for $c \in \mathcal{C}'$. Therefore also the map $L(\eta_c)$ is an equivalence since it is equivalent in the arrow category. Thus we can apply the criterion described above. □

References

[Alm74] Gert Almkvist, *The Grothendieck ring of the category of endomorphisms*, J. Algebra **28** (1974), 375–388. MR 0432738

[BFG94] D. Burghelea, Z. Fiedorowicz, and W. Gajda, *Power maps and epicyclic spaces*, J. Pure Appl. Algebra **96** (1994), no. 1, 1–14. MR 1297435

[BGN18] Clark Barwick, Saul Glasman, and Denis Nardin, *Dualizing cartesian and cocartesian fibrations*, Theory Appl. Categ. **33** (2018), Paper No. 4, 67–94. MR 3746613

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838. MR 3070515

[BGT16] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *K-theory of endomorphisms via noncommutative motives*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 1435–1465. MR 3430369

[BN13] D. Ben-Zvi and D. Nadler, *Nonlinear Traces*, ArXiv e-prints (2013).

[Cra10] J. Cranch, *Algebraic Theories and (Infinity,1)-Categories*, Ph.D. thesis, PhD Thesis, 2010, 2010.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153. MR 3450758

[GHN15] D. Gepner, R. Haugseng, and T. Nikolaus, *Lax colimits and free fibrations in ∞-categories*, arXiv:1501.02161.

[Lur09] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur16] ———, *Higher Algebra*, 2016, `www.math.harvard.edu/~lurie/papers/HA.pdf`.

[NS17] T. Nikolaus and P. Scholze, *On topological cyclic homology*, ArXiv e-prints (2017).